فصل چهارم

خصوصیات زبان‌های منظم (۱)

Properties of Regular Languages (1)

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More Properties
of
Regular Languages

We have proven

Regular languages are closed under:

Union

Concatenation

Star operation

Reverse

Namely, for regular languages L_1 and L_2 :

Union	$L_1 \cup L_2$	Regular Languages
Concatenation	$L_1 L_2$	
Star operation	$L_1{}^*$	
Reverse	$L_1{}^R$	

We will prove

Regular languages are closed under:

Complement

Intersection

Namely, for regular languages L_1 and L_2:

Complement $\overline{L_1}$

Intersection $L_1 \cap L_2$

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ Regular Languages

Complement

Theorem: For regular language L
the complement $\overline{L}$ is regular

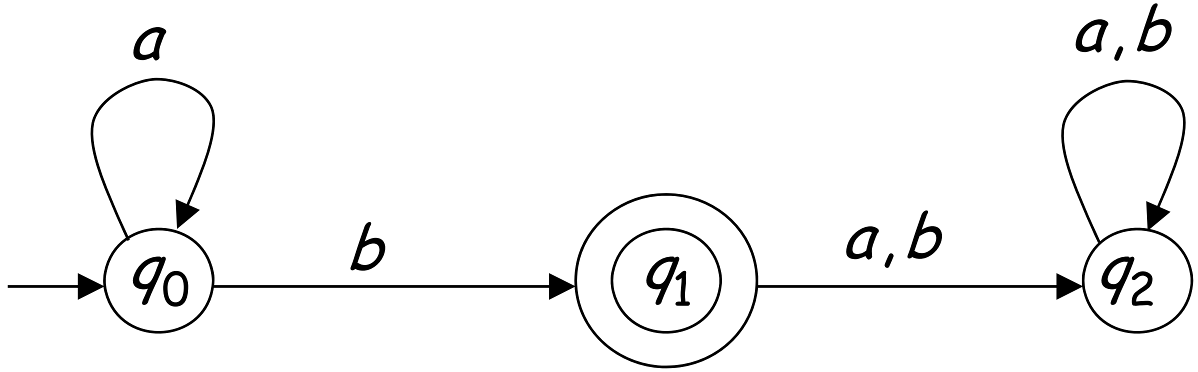
Proof: Take DFA that accepts L and make

- nonfinal states final
- final states nonfinal

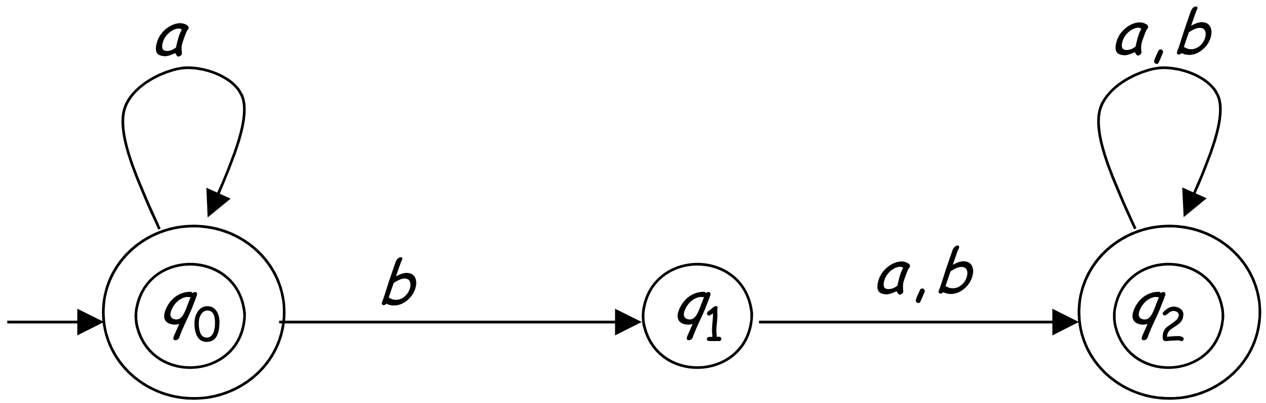
Resulting DFA accepts $\overline{L}$

Example:

$$L = L(a^* b)$$



$$\overline{L} = L(a^* + a^* b(a+b)(a+b)^*)$$



Intersection

Theorem: For regular languages L_1 and L_2
the intersection $L_1 \cap L_2$ is regular

Proof: Apply DeMorgan's Law:

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

$L_1 \,,\ L_2$ regular

$\Longrightarrow$ $\overline{L_1} \,,\ \overline{L_2}$ regular

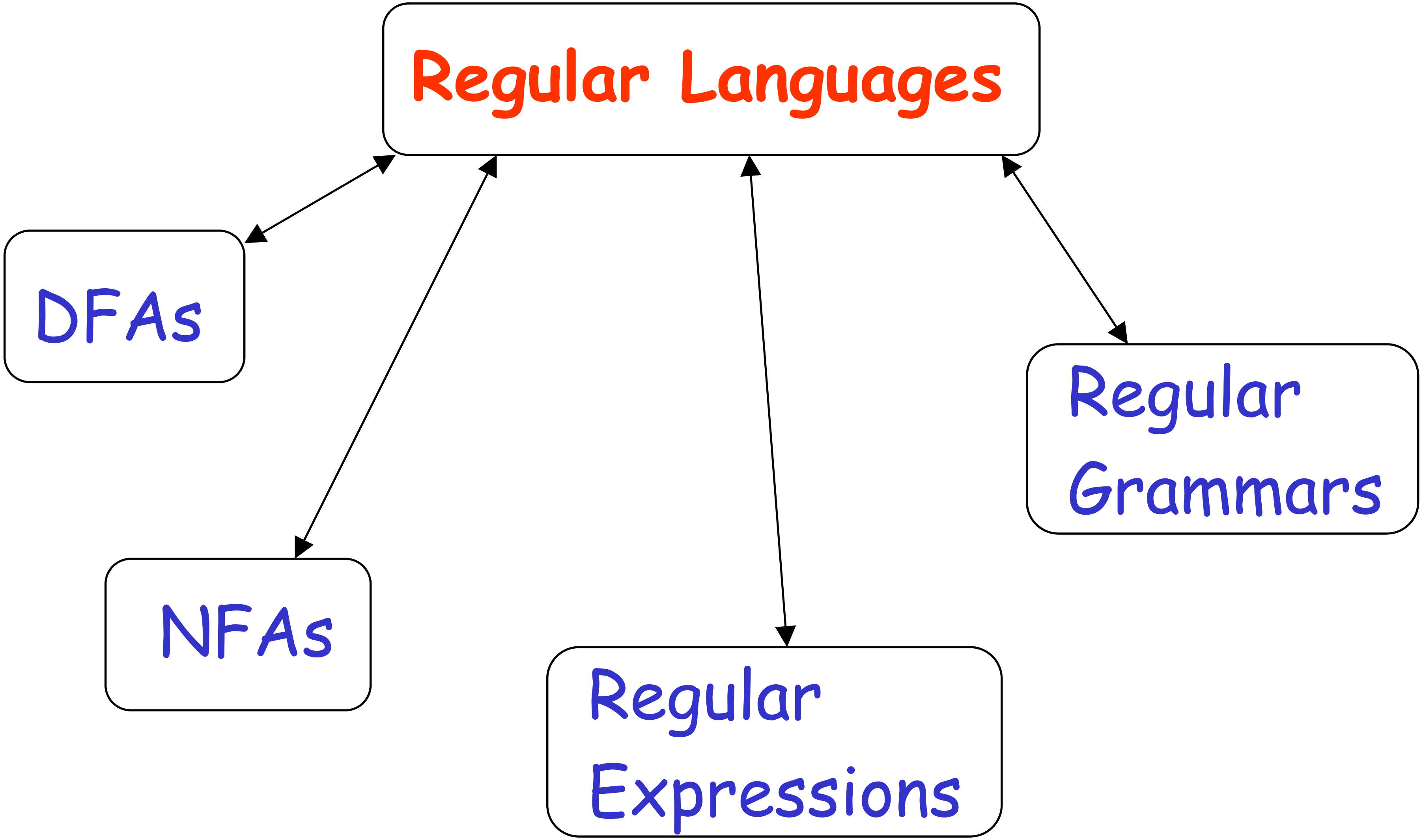
$\Longrightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Longrightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Longrightarrow$ $L_1 \cap L_2$ regular

Standard Representations
of
Regular Languages

Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

Elementary Questions

about

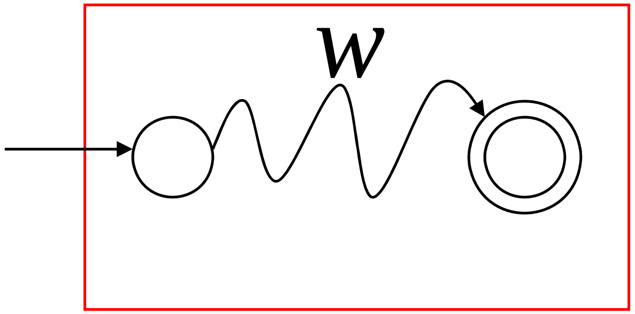
Regular Languages

Membership Question

Question: Given regular language L
and string w
how can we check if $w \in L$?

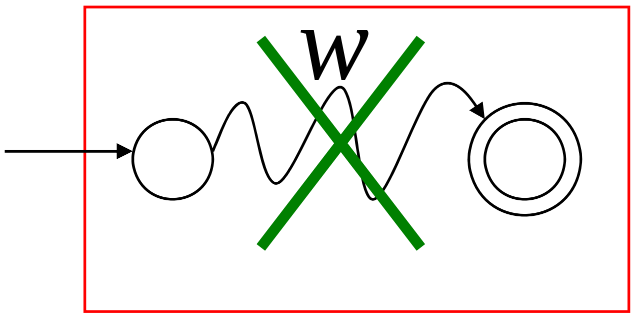
Answer: Take the DFA that accepts L
and check if w is accepted

DFA



$w \in L$

DFA

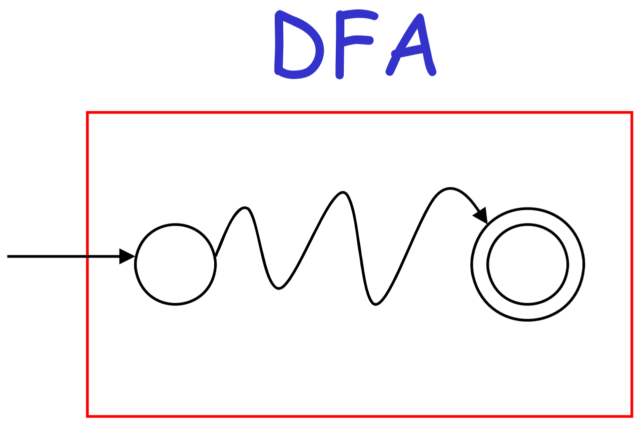


$w \notin L$

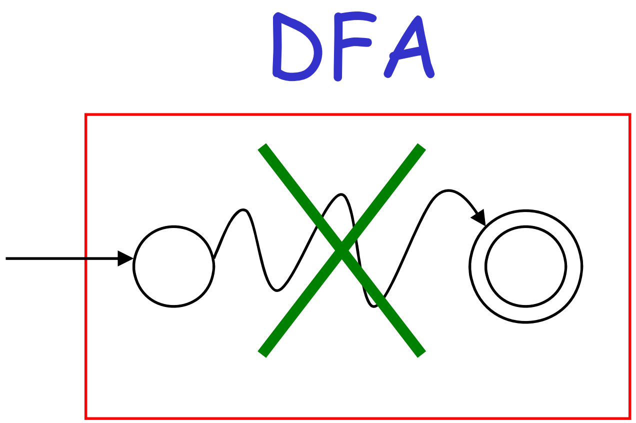
Question: Given regular language L how can we check if L is empty: $(L = \varnothing)$?

Answer: Take the DFA that accepts L

Check if there is a path from the initial state to a final state



$$L \neq \emptyset$$



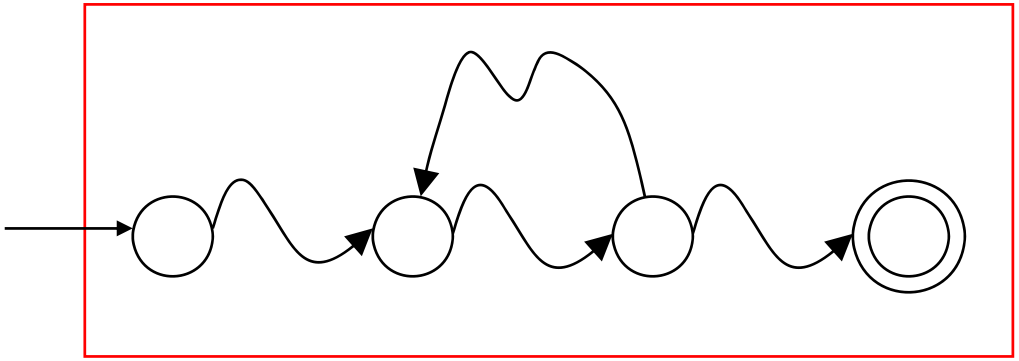
$$L = \emptyset$$

Question: Given regular language L
how can we check
if L is finite?

Answer: Take the DFA that accepts L

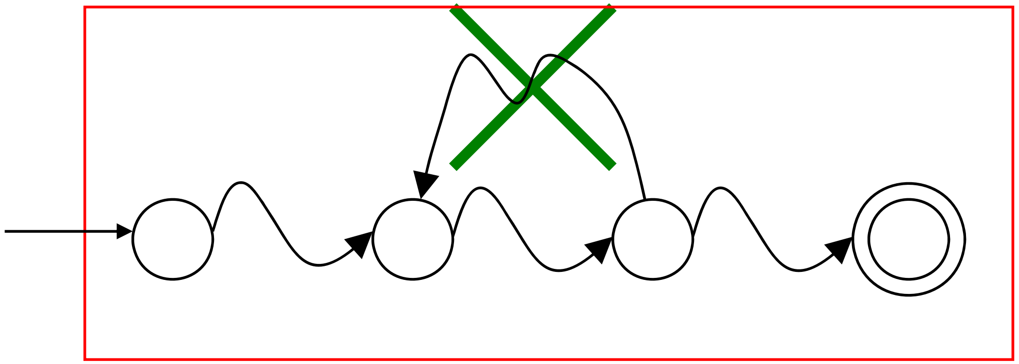
Check if there is a walk with cycle
from the initial state to a final state

DFA



L is infinite

DFA



L is finite

Question: Given regular languages L_1 and L_2 how can we check if $L_1 = L_2$?

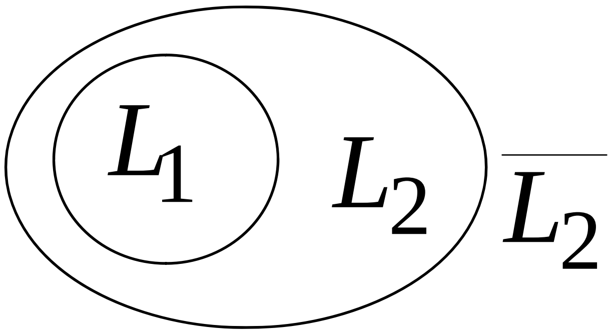
Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$

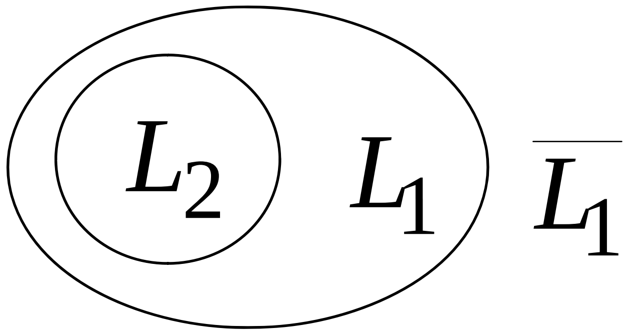
$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \varnothing$$

$$L_1 \cap \overline{L_2} = \varnothing$$

and

$$\overline{L_1} \cap L_2 = \varnothing$$





$$L_1 \subseteq L_2$$

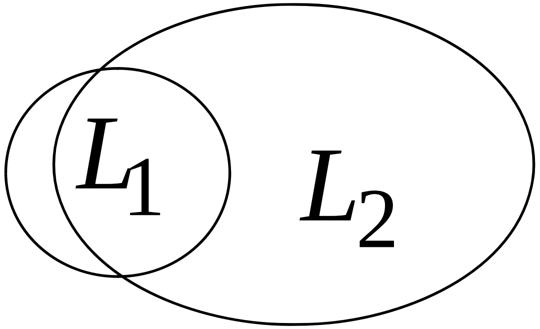
$$L_2 \subseteq L_1$$

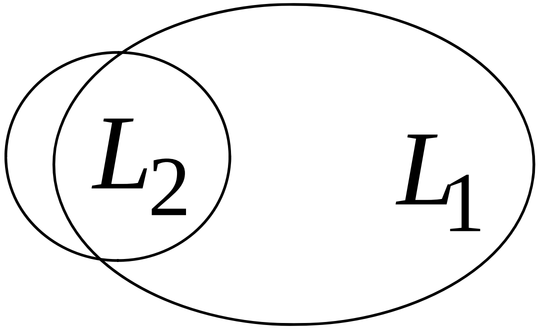
$$L_1 = L_2$$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \emptyset$$

$$\Downarrow$$

$L_1 \cap \overline{L_2} \neq \emptyset$ or $\overline{L_1} \cap L_2 \neq \emptyset$





$L_1 \not\subset L_2$ $\qquad$ $L_2 \not\subset L_1$

$$\Downarrow$$

$$L_1 \neq L_2$$

Non-regular languages

Non-regular languages

$$\{a^n b^n: \quad n \geq 0\}$$

$$\{ww^R: \quad w \in \{a,b\}^*\}$$

Regular languages

$$a^*b$$

$$b^*c+a$$

$$b+c(a+b)^*$$

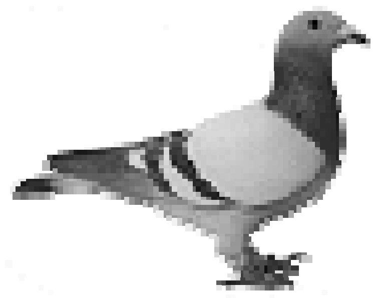
$$etc...$$

How can we prove that a language L is not regular?

Prove that there is no DFA that accepts L

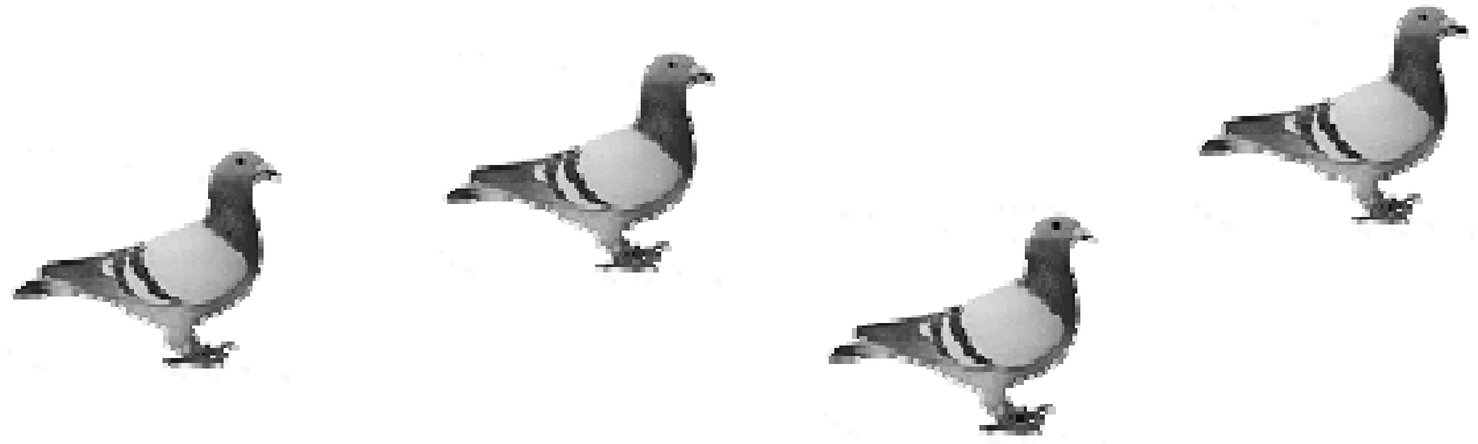
Problem: this is not easy to prove

Solution: the Pumping Lemma !!!

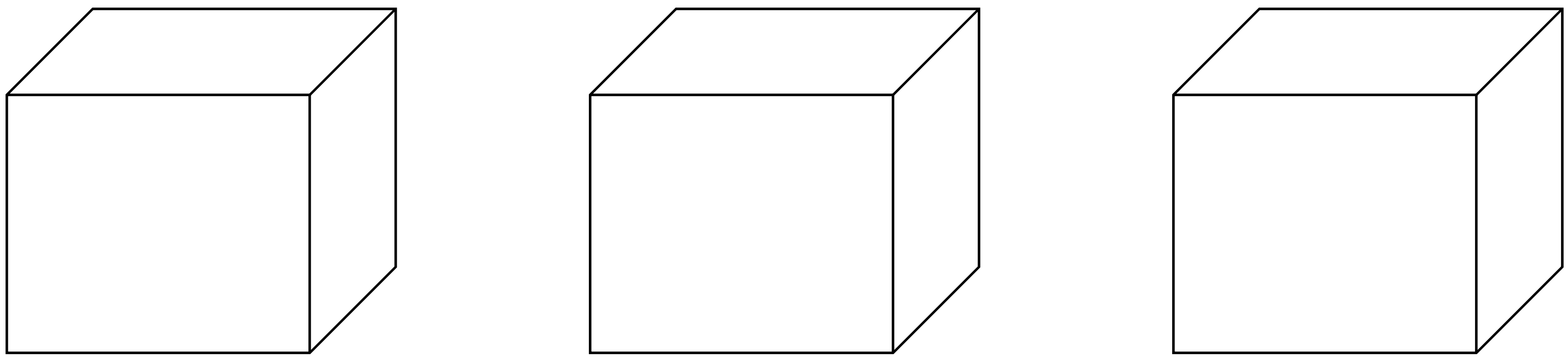


The Pigeonhole Principle

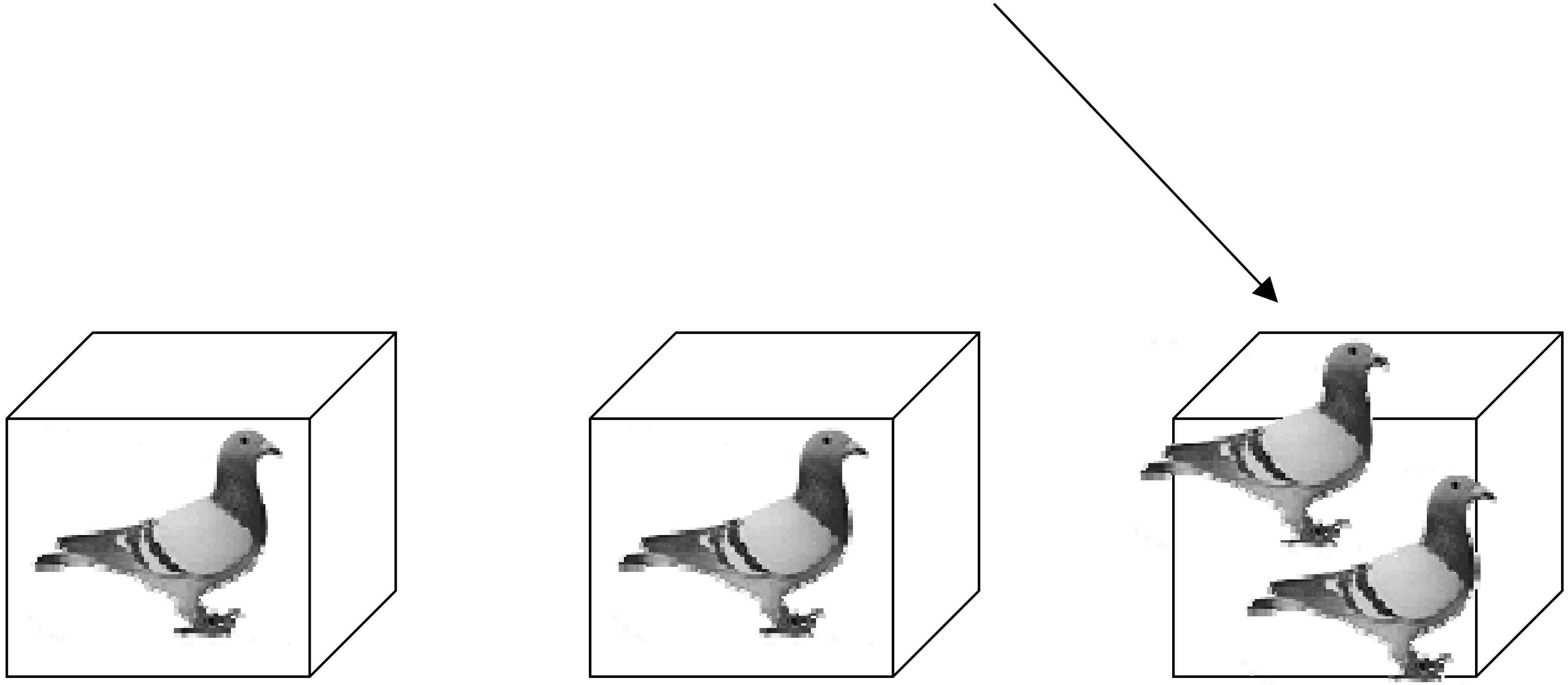
4 pigeons



3 pigeonholes

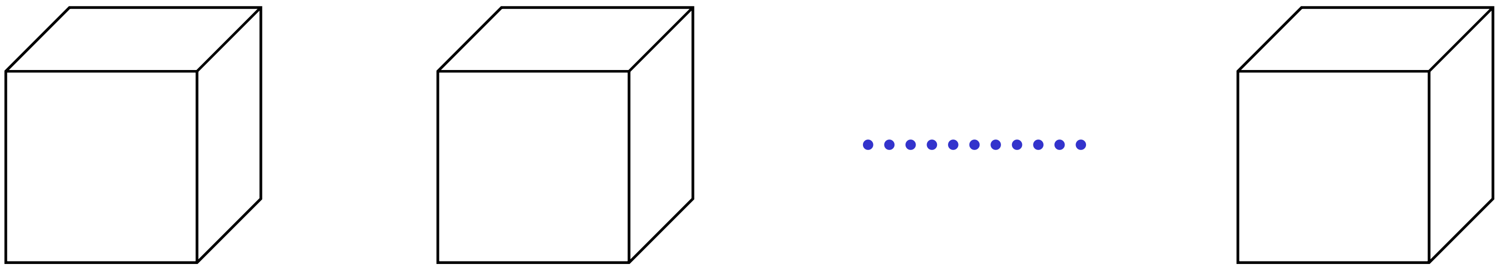


A pigeonhole must contain at least two pigeons



n pigeons

m pigeonholes $\quad n > m$



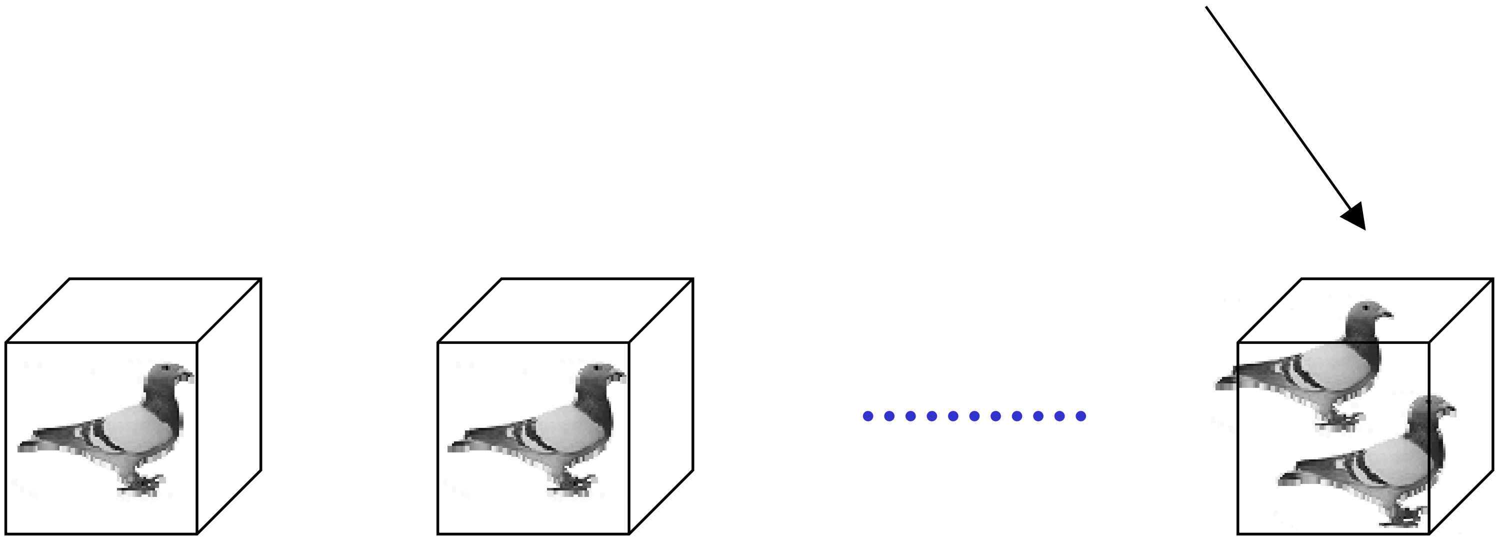
The Pigeonhole Principle

n pigeons

m pigeonholes

$$n > m$$

There is a pigeonhole with at least 2 pigeons

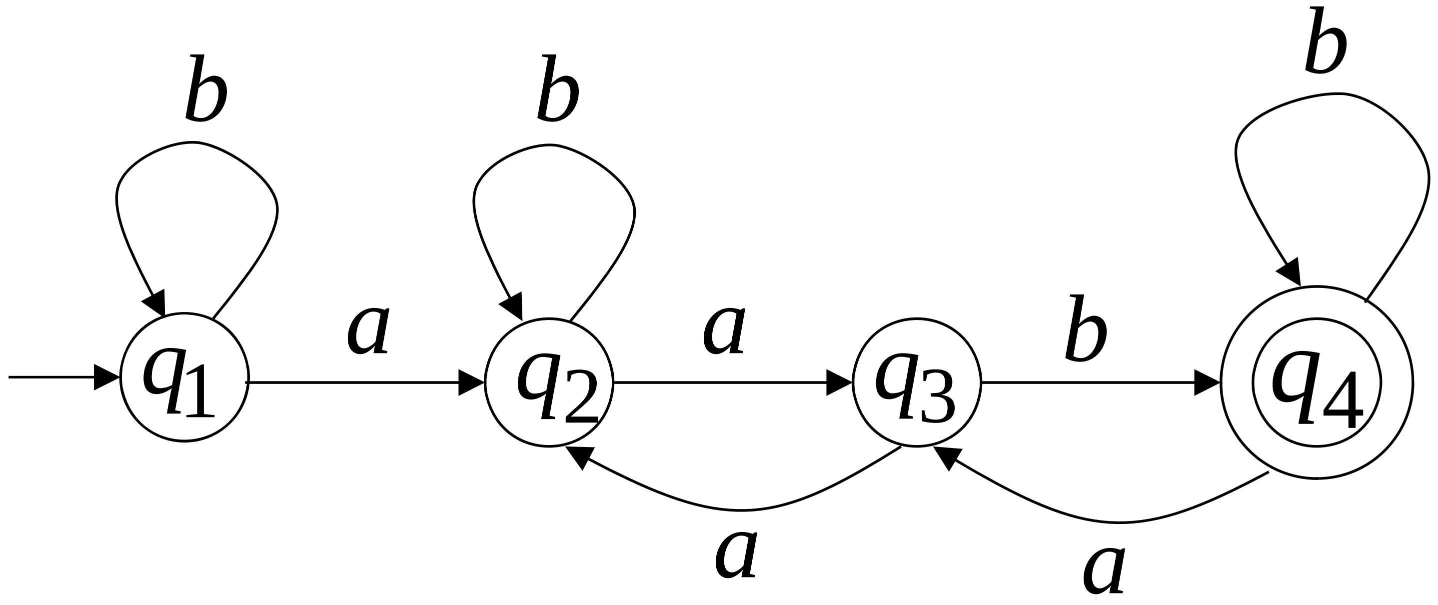


The Pigeonhole Principle

and

DFAs

DFA with 4 states

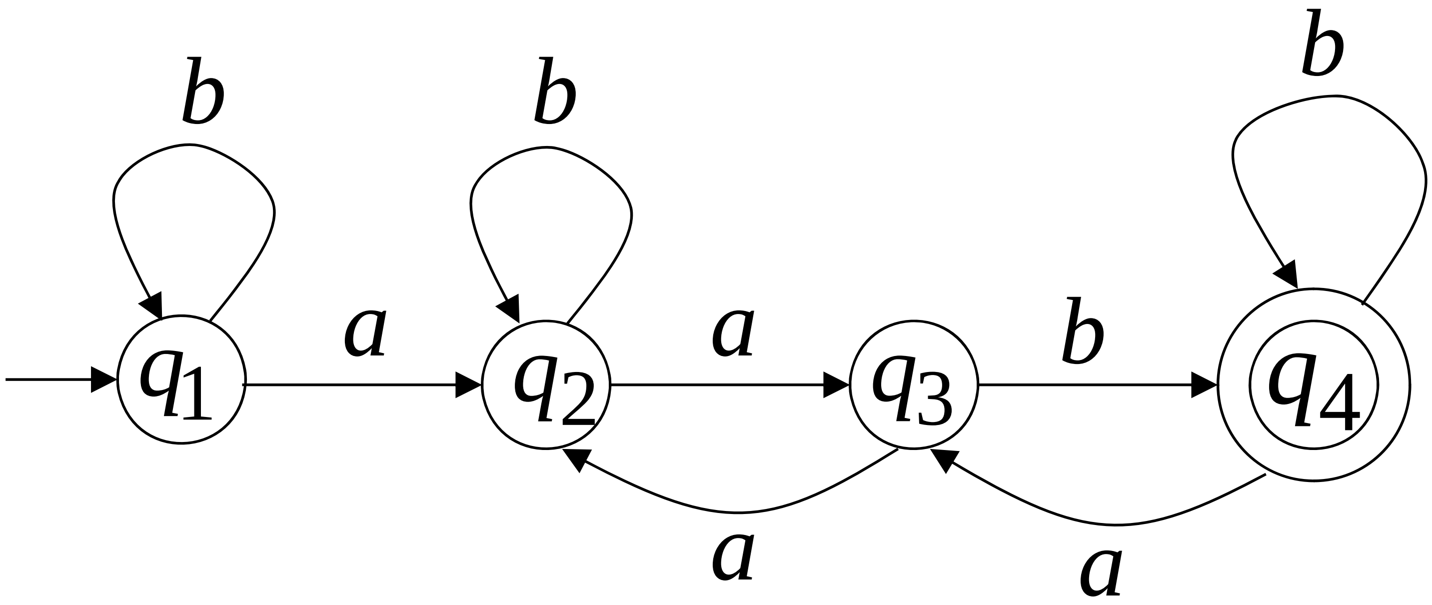


In walks of strings: a

aa

aab

no state is repeated



In walks of strings: $aabb$

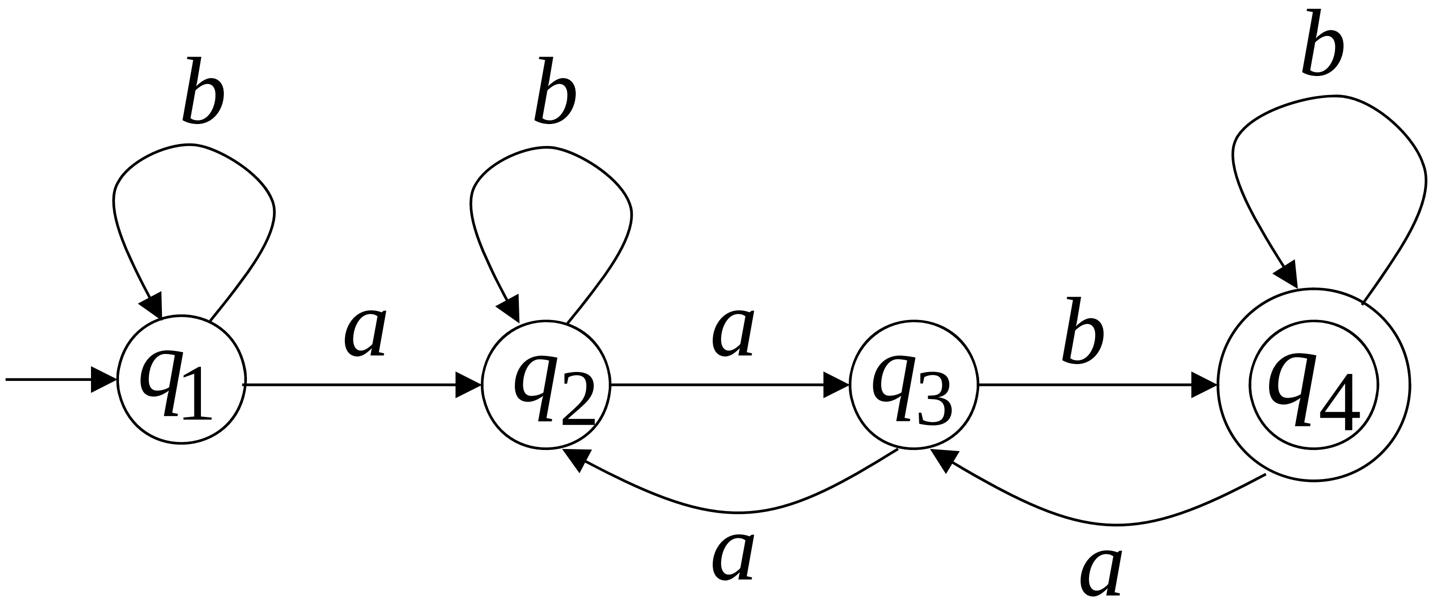
$bbaa$

$abbabb$

$abbbabbabb...$

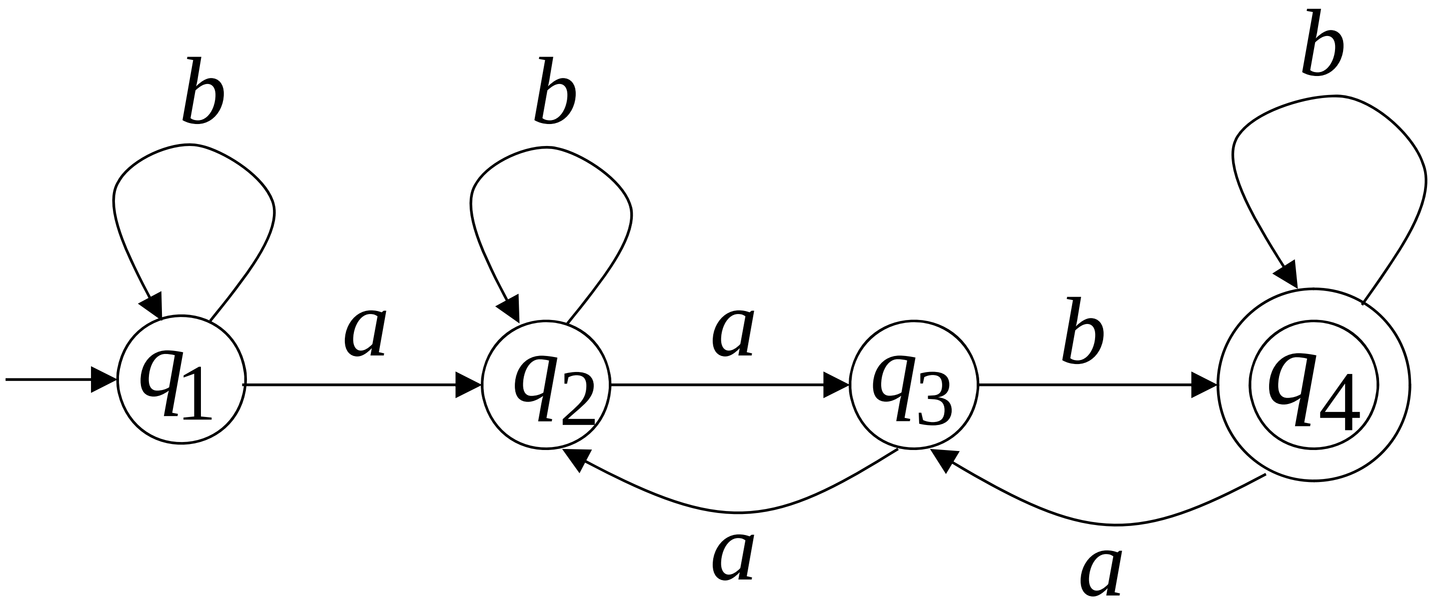
a state

is repeated



If the walk of string w has length $|w| \geq 4$

then a state is repeated

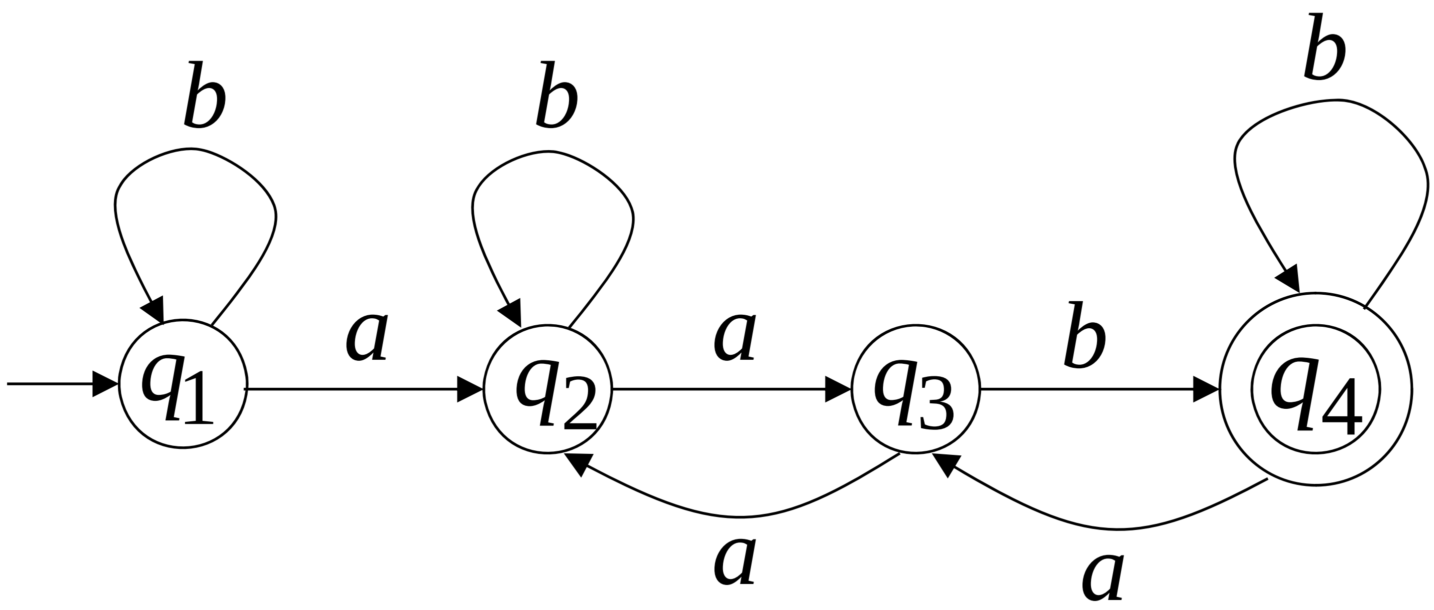


Pigeonhole principle for any DFA:

If in a walk of a string w

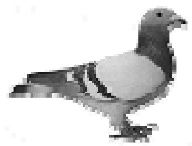
transitions $\geq$ states of DFA

then a state is repeated

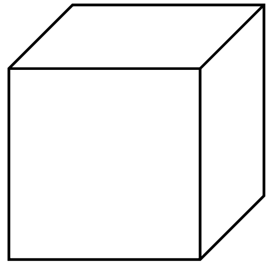


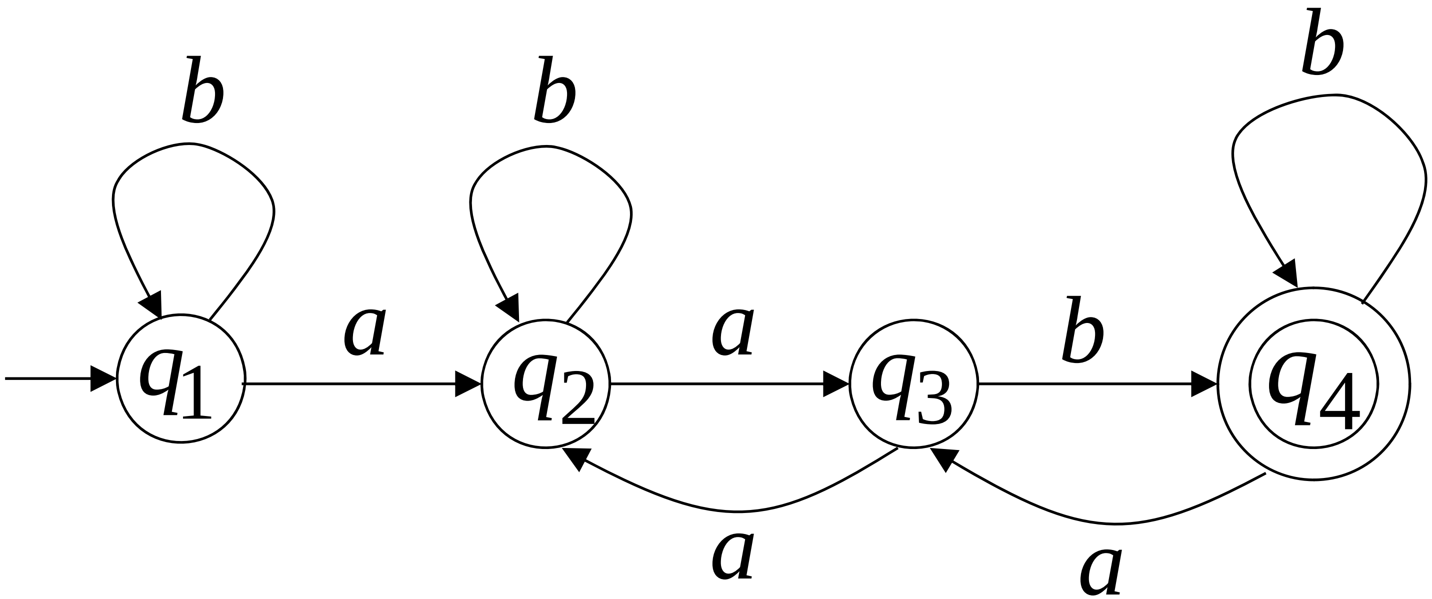
In other words for a string w:

$\xrightarrow{a}$ transitions are pigeons



q states are pigeonholes



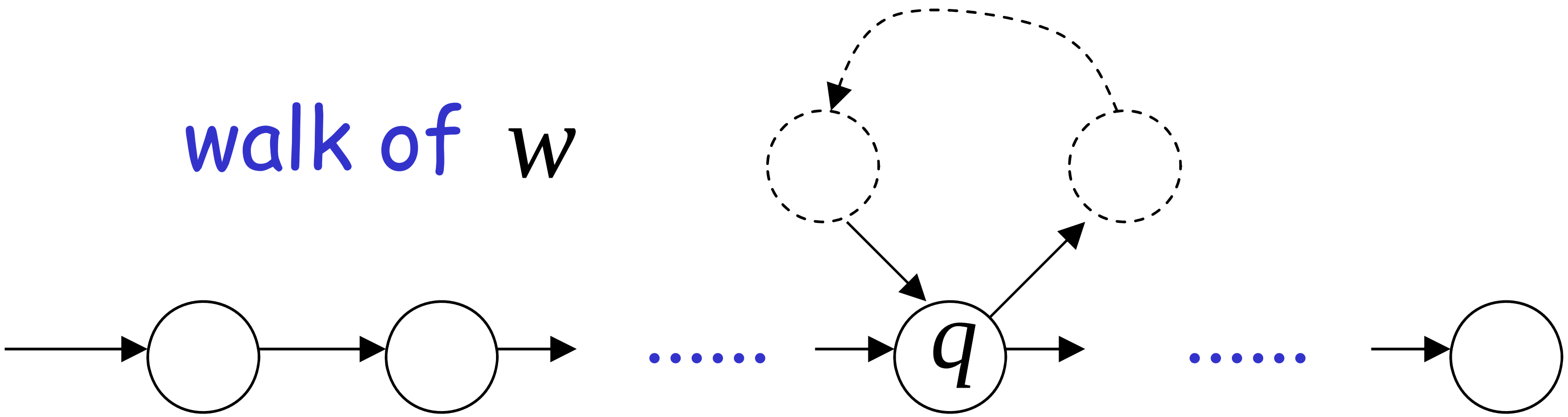


In general:

A string w has length $\geq$ number of states

A state q must be repeated in the walk w

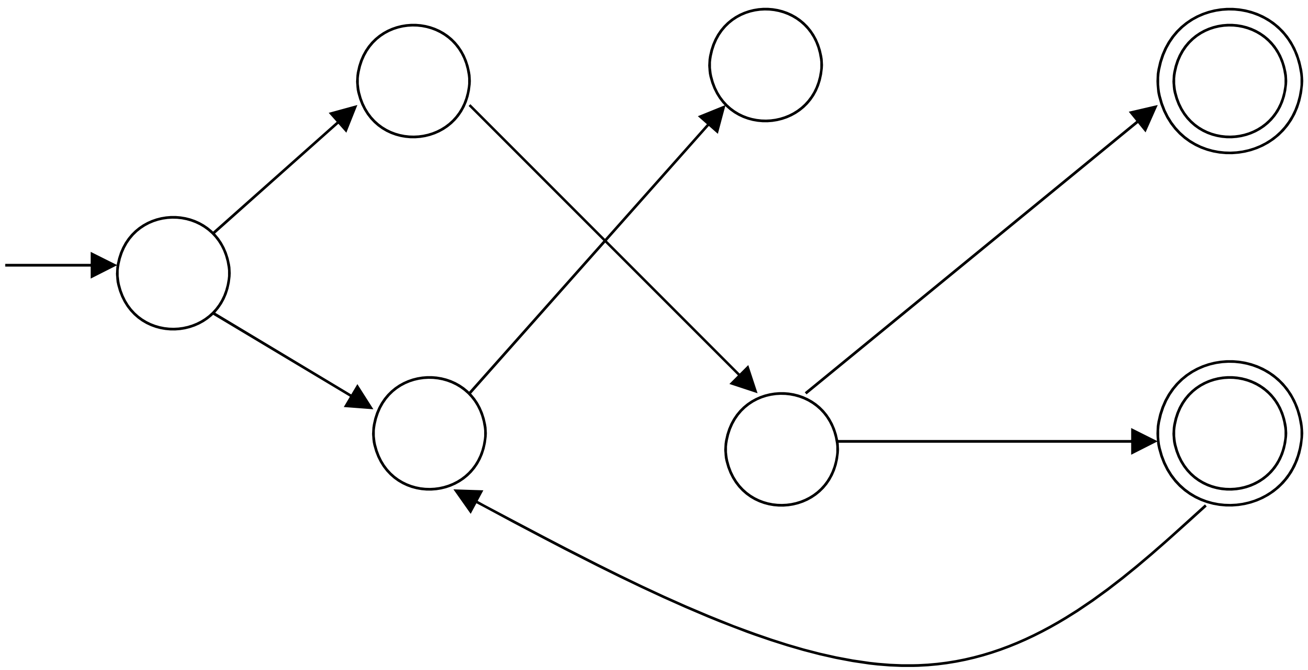
walk of w



The Pumping Lemma

Take an infinite regular language L

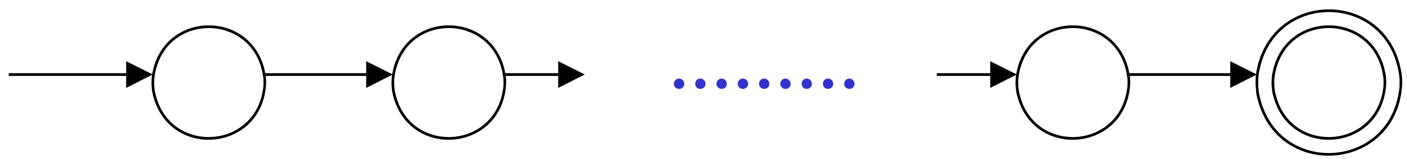
DFA that accepts L



m
states

Take string w with $w \in L$

There is a walk with label w:

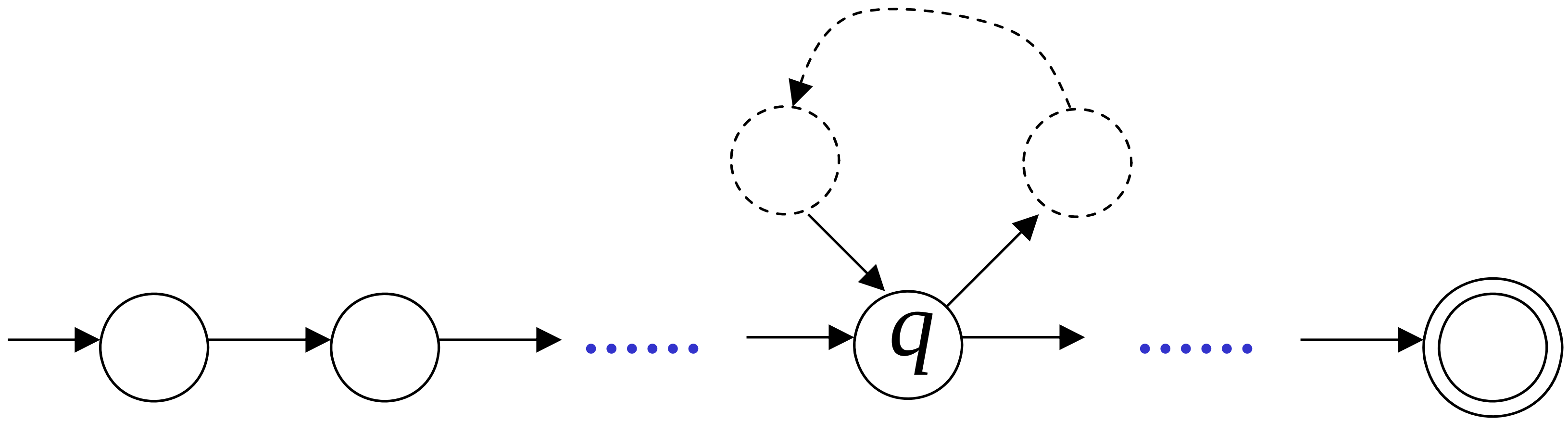


walk w

If string w has length $|w| \geq m$ number of states

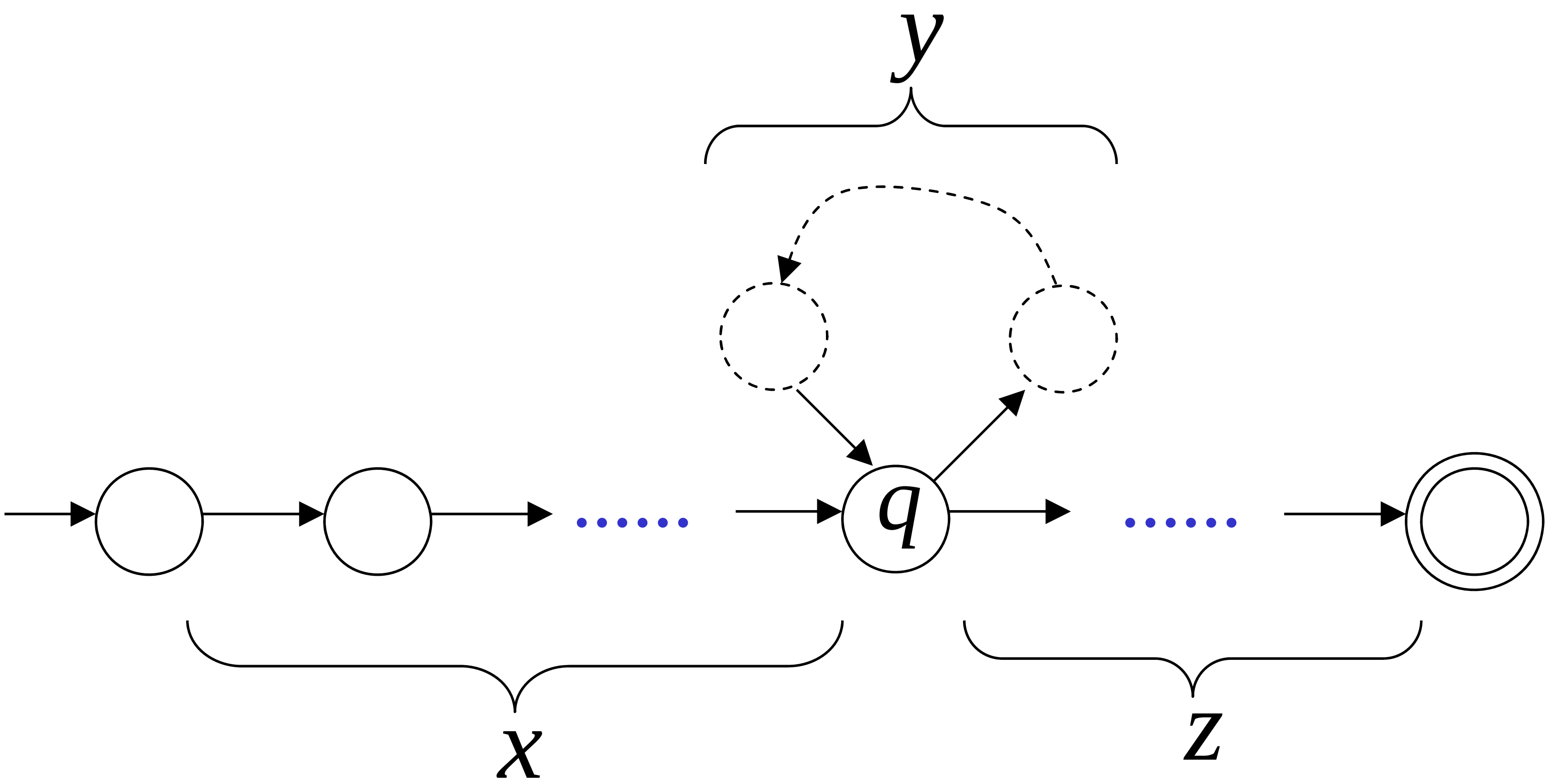
then, from the pigeonhole principle:

a state q is repeated in the walk w



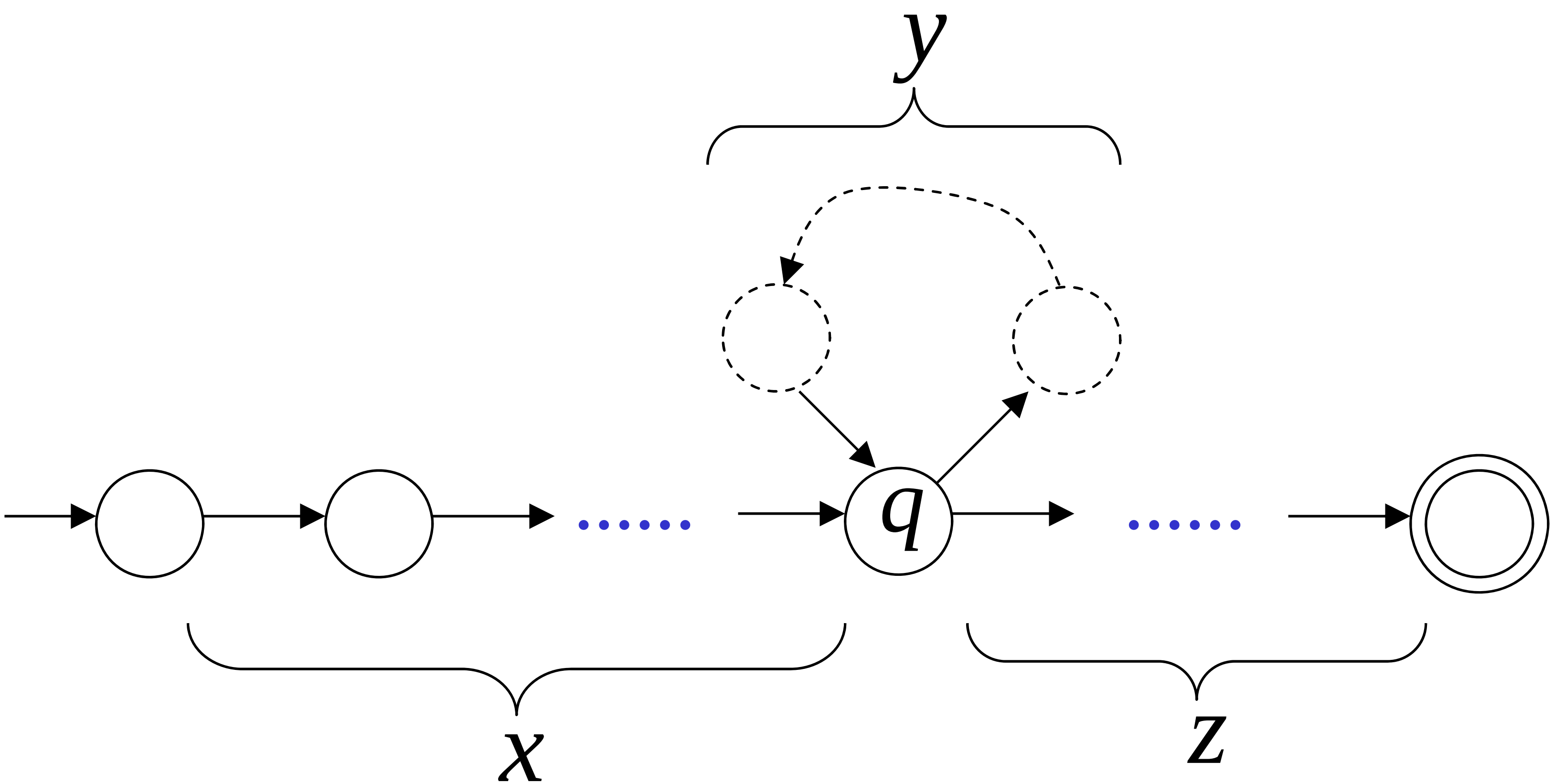
walk w

Write $w = x\ y\ z$

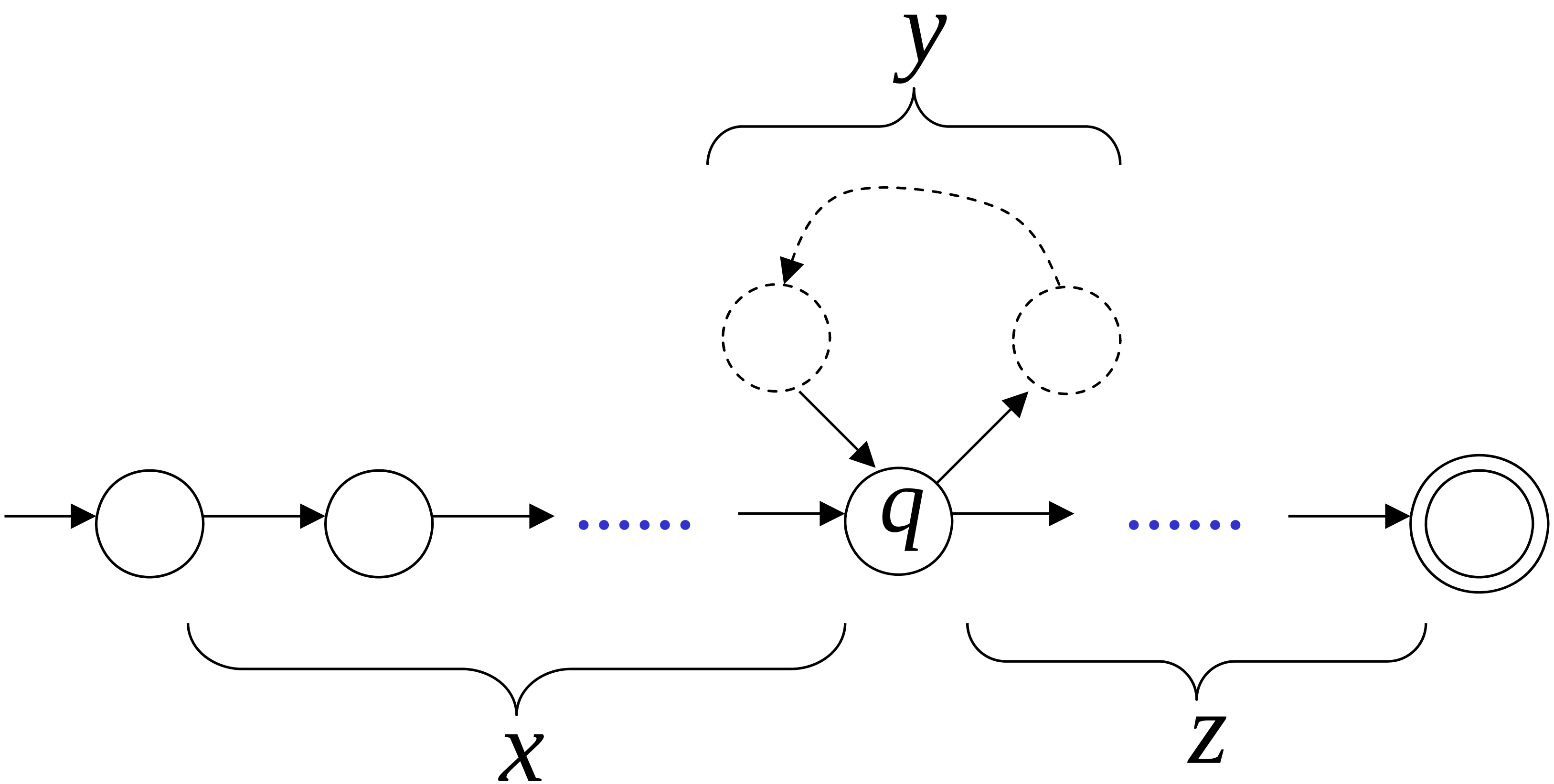


Observations: length $|x\,y| \leq m$ number of states

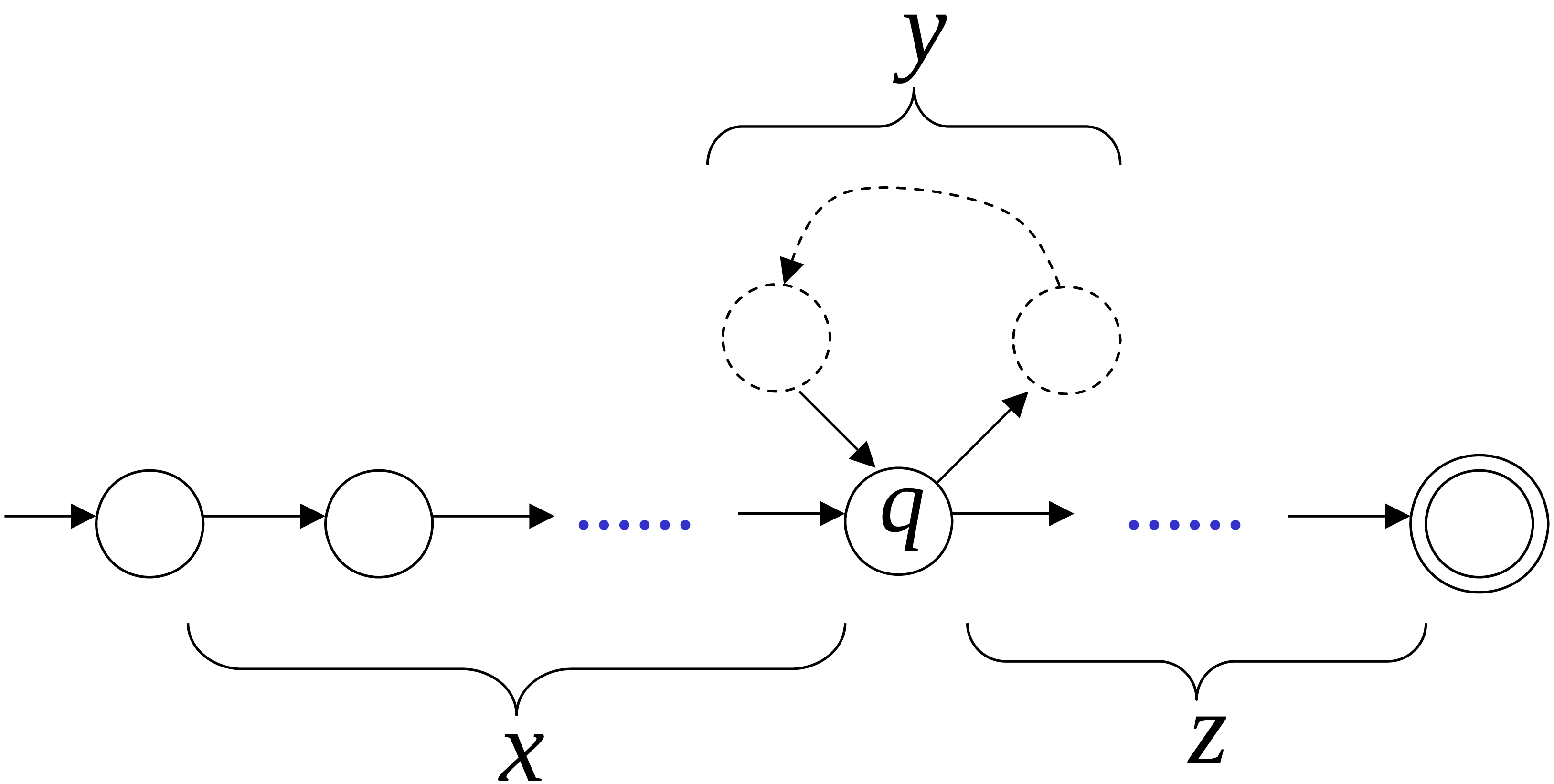
length $|y| \geq 1$



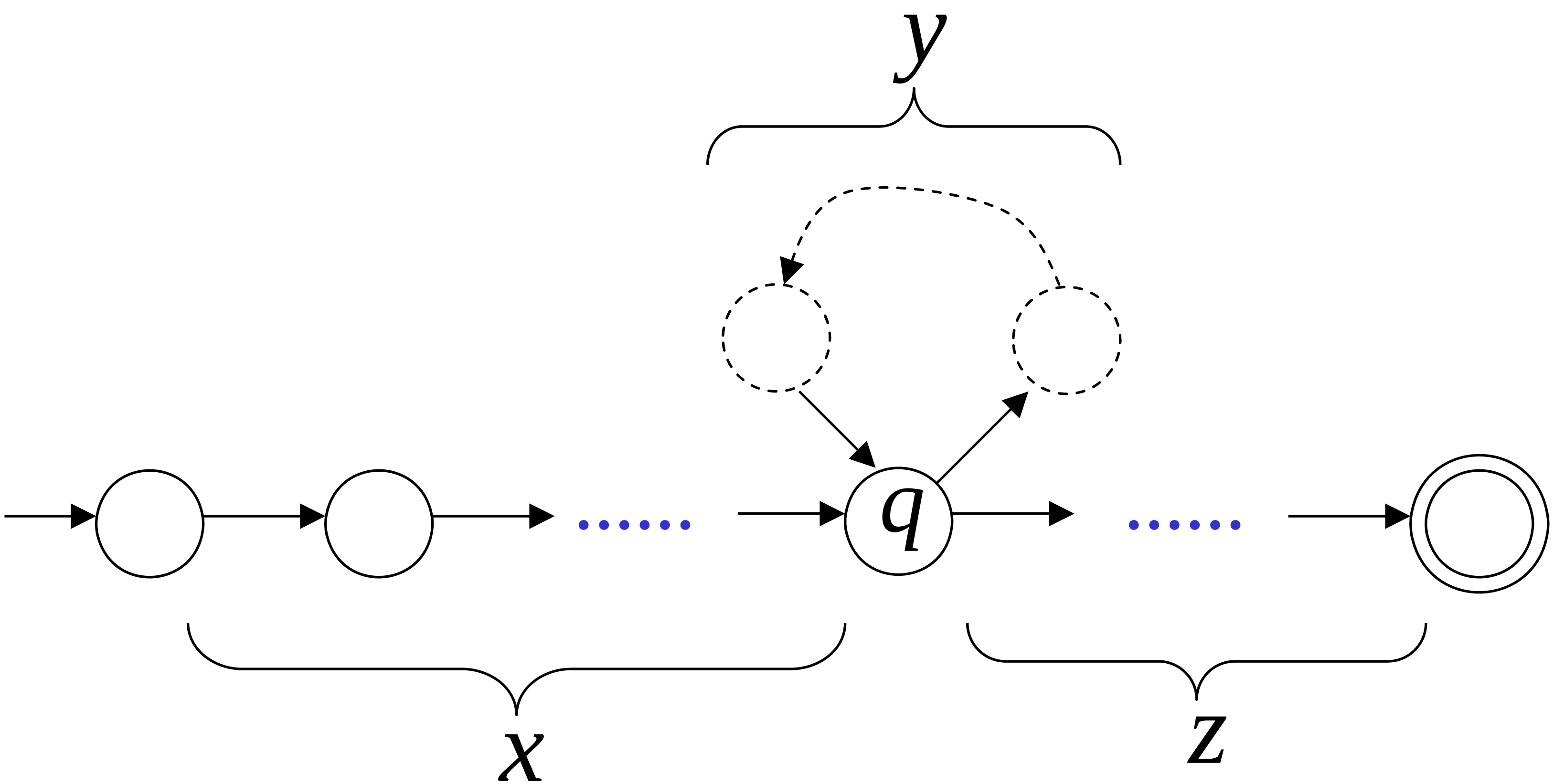
Observation: The string $x\,z$ is accepted



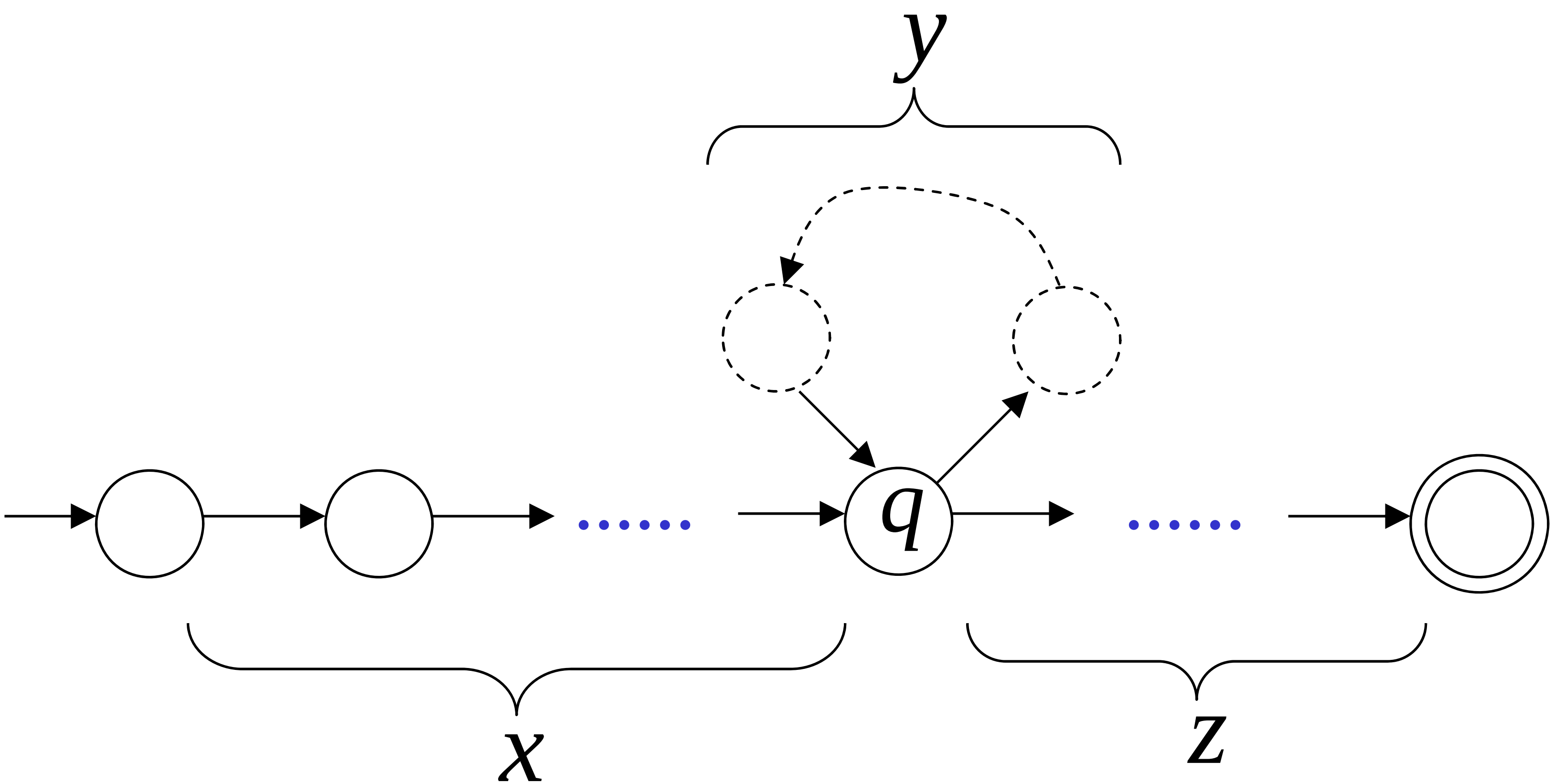
Observation: The string $x\ y\ y\ z$ is accepted



Observation: The string $x\ y\ y\ y\ z$ is accepted



In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\, y\, z$
- with $|x\, y| \leq m$ and $|y| \geq 1$
- such that: $x\, y^i\, z \in L \qquad i = 0, 1, 2, \ldots$

Applications

of

the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$

is not regular

Proof: Use the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Assume for contradiction
that L is a regular language

Since L is infinite
we can apply the Pumping Lemma

$$L = \{a^n b^n : n \geq 0\}$$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

Example: pick $w = a^m b^m$

Write: $a^m b^m = x\ y\ z$

From the Pumping Lemma

it must be that: length $|\,x\ y\,| \leq m, \quad |\,y\,| \geq 1$

Therefore: $$a^m b^m = \underbrace{\overbrace{\underbrace{a \ldots a}_{x}\underbrace{a \ldots a}_{y}\ldots a}^{m}\overbrace{b \ldots b}^{m}}$$

$$a^m b^m = \underbrace{a \ldots a}_{x}\underbrace{a \ldots a}_{y}\underbrace{\ldots ab \ldots b}_{z}$$

$y = a^k, \quad k \geq 1$

We have: $x\,y\,z = a^m b^m$ $y = a^k, \quad k \geq 1$

From the Pumping Lemma: $x\,y^i\,z \in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^2\,z \in L$

$$x\,y^2\,z = x\,y\,y\,z = a^{m+k} b^m \in L$$

Therefore: $a^{m+k}b^m \in L$

BUT: $L = \{a^n b^n : n \geq 0\}$

$\Downarrow$

$a^{m+k}b^m \notin L$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages $\{a^n b^n : \ n \geq 0\}$

Regular languages

a^*b

$b^*c + a$

$b + c(a+b)^*$

etc...