Non-Newtonian Fluid Mechanics

(Part - IX)

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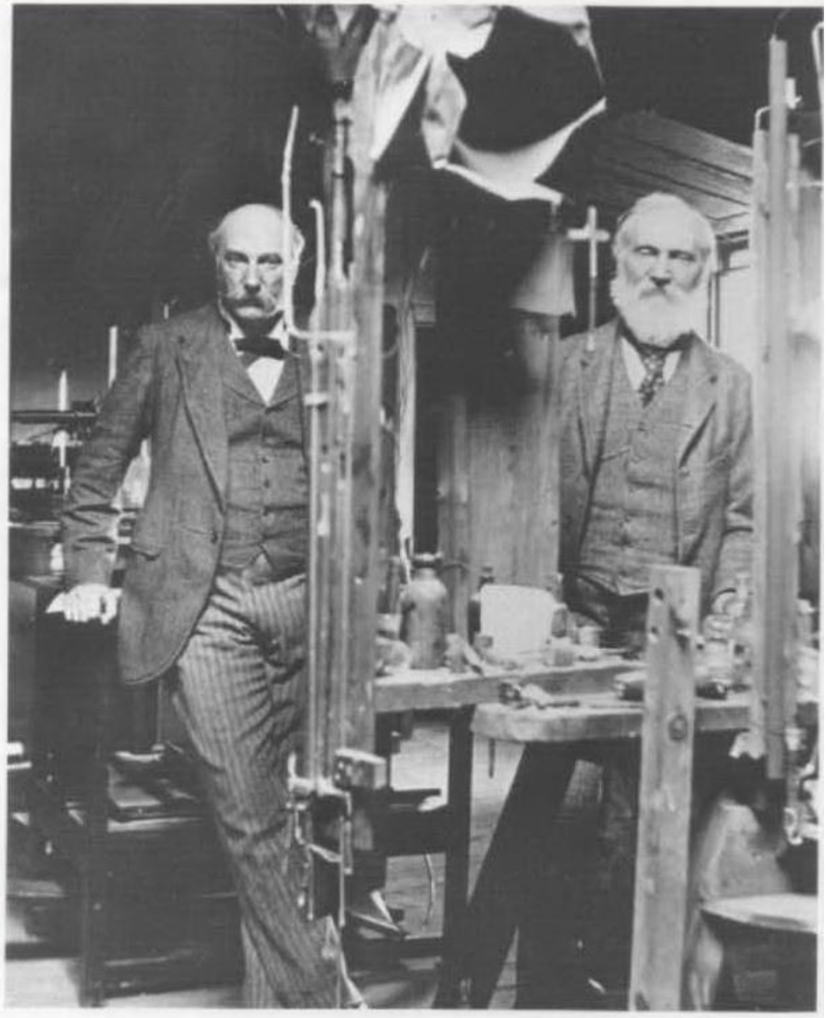
Modeling!

"I am never content until I have constructed a mechanical model of the subject I am studying....
I often say that when you can measure what you are speaking about and express it in numbers, you know something about it;
but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meager and unsatisfactory kind",

1897 William Thomson (Baron Kelvin),
A Dictionary of Scientific Quotations (Oxford)



Constitutive Equations for Polymeric Liquids

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CONSTITUTIVE EQUATIONS FOR POLYMERIC LIQUIDS

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INTRODUCTION

This review addresses the origins, uses, and evaluation of constitutive equations for the stress tensor of polymeric liquids. The continuum aspects of the subject up to about 1986 were summarized by Bird et al (1987a), and the molecular aspects by Bird et al (1987b); these two textbooks will be referred to as DPL-1 and DPL-2. Bird & Öttinger (1992) review advances in molecular theory from 1986 to 1991. Here we put into perspective those aspects of the subject that are of primary concern in fluid dynamics, with extra emphasis on noteworthy advances of the past decade. A comparison of this review with one prepared by Bird (1976) nearly two decades ago will show that much progress has been made in this field and that there has been a considerable shift in emphasis, largely because of increased computational capability and the influence of developments in kinetic theory.

To solve fluid dynamics problems we use the equations of continuity, motion, and energy:

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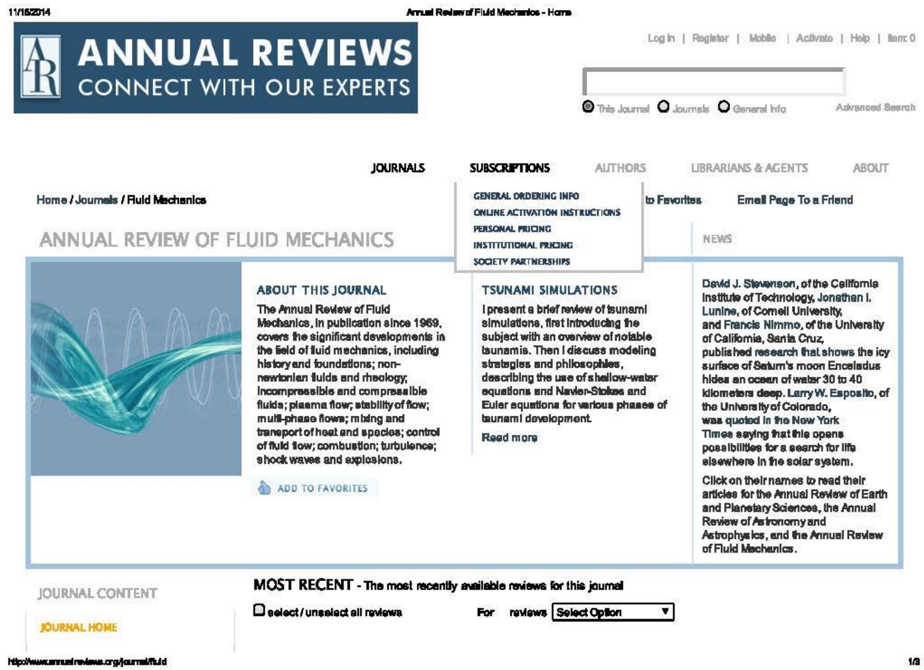
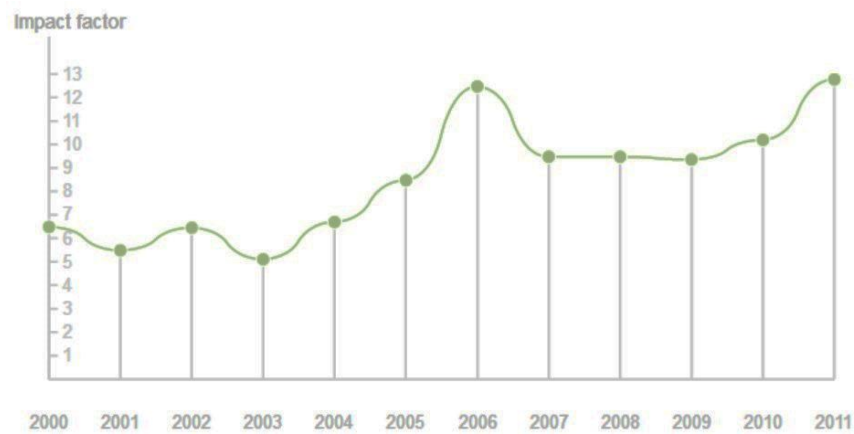




Constitutive Equations

Methods for Developing Constitutive Equations

1. Purely empirical models
2. Using mathematical expansions
3. Developing equations of a rather general nature that apply only within certain well-defined classes of flows
4. Molecular theory
5. Using various extensions of the thermodynamics of irreversible processes

Constitutive equations may be obtained by replacing the time derivatives by convected derivatives in linear viscoelastic models. The obtained models called quasi-linear models.

Jeffreys linear Model:

$$\tau+\lambda_1\tau=\eta\left(\gamma+\lambda_2\gamma\right)$$

Oldroyd-A Model

$$\tau+\lambda_1\tau^{(1)}=\eta\left(\gamma^{(1)}+\lambda_2\gamma^{(2)}\right) \longrightarrow \Psi_1>0\ \&\ \Psi_2=-\Psi_1$$

Oldroyd-B Model

$$\tau+\lambda_1\tau_{(1)}=\eta\left(\gamma_{(1)}+\lambda_2\gamma_{(2)}\right) \longrightarrow \Psi_1>0\ \&\ \Psi_2=0$$

Maxwell Model: $\tau+\lambda\tau=\eta\gamma$ $\longrightarrow$ **Upper Convected Maxwell Model (UCM):** $\tau+\lambda\tau_{(1)}=\eta\gamma_{(1)}$

Retarded Expansion of Motion Models

The retarded expansion of motion models are derived based on the Taylor expansion series of shear rates terms around the Newtonian behavior.

These models are suitable for low speed viscoelastic flows. It is possible to show that there are exact for creeping motions.

Second Order Fluid

$$\boldsymbol{\tau} = \underbrace{b_1 \boldsymbol{\gamma}_{(1)} + b_2 \boldsymbol{\gamma}_{(2)} + b_{11}\{\boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)}\}} + b_3 \boldsymbol{\gamma}_{(3)} + b_{12}\{\boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(2)} + \boldsymbol{\gamma}_{(2)} \cdot \boldsymbol{\gamma}_{(1)}\} + b_{1:11}(\boldsymbol{\gamma}_{(1)} : \boldsymbol{\gamma}_{(1)})\boldsymbol{\gamma}_{(1)} + \cdots \quad (1)$$

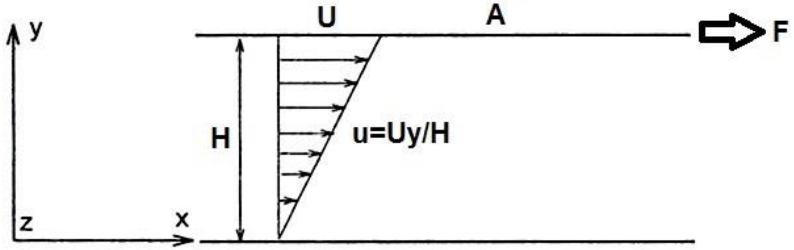
Third Order Fluid

Viscometric Functions of the Third-Order Fluid

Determining the viscometric functions of the third-order fluid in steady shear flow:

In steady shear flow, we have:

$$u = u(y),\ v = 0\ \ \&\ w = 0 \qquad (2)$$



The shear rate and velocity gradient of this flow can be expressed as follows:

$$\dot{\gamma} = \nabla\mathbf{V} + \nabla\mathbf{V}^T = \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 2\frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\gamma}_{xy} \qquad (3a)$$

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\gamma}_{xy} \qquad (3b)$$

Viscometric Functions of the Third-Order Fluid

Now, we should determine the terms of Eq. (1) using Eqns. (2) and (3):

$$
\begin{aligned}
&\boldsymbol{\gamma}_{(1)} = \nabla\mathbf{V} + \nabla\mathbf{V}^T = \dot{\boldsymbol{\gamma}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}_{xy} \\
&\boldsymbol{\gamma}_{(2)} = \frac{D\boldsymbol{\gamma}_{(1)}}{Dt} - \left\{ \nabla\mathbf{V}^T \boldsymbol{\gamma}_{(1)} + \boldsymbol{\gamma}_{(1)} \nabla\mathbf{V} \right\} = \frac{\partial \boldsymbol{\gamma}_{(1)}}{\partial t} + u\frac{\partial \boldsymbol{\gamma}_{(1)}}{\partial x} + v\frac{\partial \boldsymbol{\gamma}_{(1)}}{\partial y} + w\frac{\partial \boldsymbol{\gamma}_{(1)}}{\partial z} - \left\{ \nabla\mathbf{V}^T \boldsymbol{\gamma}_{(1)} + \boldsymbol{\gamma}_{(1)} \nabla\mathbf{V} \right\} \\
&\boldsymbol{\gamma}_{(2)} = -\left\{ \nabla\mathbf{V}^T \boldsymbol{\gamma}_{(1)} + \boldsymbol{\gamma}_{(1)} \nabla\mathbf{V} \right\} = -\left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \dot{\gamma}_{xy}^{\,2} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}_{xy}^{\,2} \\
&\boldsymbol{\gamma}_{(3)} = \frac{D\boldsymbol{\gamma}_{(2)}}{Dt} - \left\{ \nabla\mathbf{V}^T \boldsymbol{\gamma}_{(2)} + \boldsymbol{\gamma}_{(2)} \nabla\mathbf{V} \right\} = -\left\{ \nabla\mathbf{V}^T \boldsymbol{\gamma}_{(2)} + \boldsymbol{\gamma}_{(2)} \nabla\mathbf{V} \right\} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&\boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}_{xy}^{\,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}_{xy}^{\,2} \qquad \& \qquad \boldsymbol{\gamma}_{(1)} : \boldsymbol{\gamma}_{(1)} = tr\left\{ \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)} \right\} = 2\dot{\gamma}_{xy}^{\,2} \\
&\boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(2)} + \boldsymbol{\gamma}_{(2)} \cdot \boldsymbol{\gamma}_{(1)} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \dot{\gamma}_{xy}^{\,3} = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\gamma}_{xy}^{\,3}
\end{aligned}
\tag{4}
$$

Viscometric Functions of the Third-Order Fluid

By substituting Eq. (4) into the Eq. (1), we have

$$\begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(b_1 \dot{\gamma}_{xy} + 2\left(b_{1:11} - b_{12}\right) \dot{\gamma}_{xy}{}^3 \right) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \left(-2b_2\right) \dot{\gamma}_{xy}{}^2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} b_{11} \dot{\gamma}_{xy}{}^2 \quad (5)$$

Based on Eq. (5), we can determine the component of stress as follows:

$$\begin{aligned} &\tau_{xx} = \left(b_{11} - 2b_2\right) \dot{\gamma}_{xy}{}^2 \qquad \tau_{yy} = b_{11} \dot{\gamma}_{xy}{}^2 \qquad \tau_{zz} = 0 \\ &\tau_{xy} = \tau_{yx} = b_1 \dot{\gamma}_{xy} + 2\left(b_{1:11} - b_{12}\right) \dot{\gamma}_{xy}{}^3 \quad \tau_{xz} = \tau_{zx} = 0 \quad \tau_{yz} = \tau_{zy} = 0 \end{aligned} \quad (6)$$

Finally, the viscometric functions are obtained as follows:

$$\begin{cases} \eta = \tau_{xy} / \dot{\gamma}_{xy} \\ \Psi_1 = \left(\tau_{xx} - \tau_{yy}\right) / \dot{\gamma}_{xy}{}^2 \\ \Psi_2 = \left(\tau_{yy} - \tau_{zz}\right) / \dot{\gamma}_{xy}{}^2 \end{cases} \longrightarrow \begin{cases} \eta = b_1 + 2\left(b_{1:11} - b_{12}\right) \dot{\gamma}_{xy}{}^2 \\ \Psi_1 = -2b_2 \\ \Psi_2 = b_{11} \end{cases} \quad (7)$$

The second Order Fluid:

The second order fluid is the first deviation from the Newtonian behavior:

$$\boldsymbol{\tau} = b_1 \boldsymbol{\gamma}_{(1)} + b_2 \boldsymbol{\gamma}_{(2)} + b_{11} \left\{ \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)} \right\} \tag{8}$$

This model is considered as a quasilinear model because its viscometric functions are constant (independent from shear rate):

$$\begin{cases} \eta = b_1 \\ \Psi_1 = -2b_2 \\ \Psi_2 = b_{11} \end{cases} \tag{9}$$

The following forms of this constitutive equation is sometimes used to model the non-Newtonian fluid flows:

$$\begin{gathered} \boldsymbol{\tau} = \eta_0 \boldsymbol{\gamma}_{(1)} - \frac{1}{2} \Psi_{1,0} \boldsymbol{\gamma}_{(2)} + \Psi_{2,0} \left\{ \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)} \right\} \\ \text{or} \\ \boldsymbol{\tau} = \eta_0 \boldsymbol{\gamma}^{(1)} - \frac{1}{2} \Psi_{1,0} \boldsymbol{\gamma}^{(2)} + \left(\Psi_{1,0} + \Psi_{2,0} \right) \left\{ \boldsymbol{\gamma}^{(1)} \cdot \boldsymbol{\gamma}^{(1)} \right\} \end{gathered} \tag{10}$$

where η_0, $\Psi_{1,0}$ and $\Psi_{2,0}$ are the viscosity and the coefficients of the first and second normal stress differences, respectively.

The Creeping Flow and Modified Pressure

The modified pressure is defined based on the following combination of gravitational acceleration and pressure:

$$\nabla \mathscr{P} = \nabla p - \rho \boldsymbol{g} \tag{11}$$

This term is used to simplify the governing equations. As mentioned before, the governing equations of non-Newtonian fluids are as follows:

$$\begin{aligned} &\nabla \cdot \mathbf{v} = 0 \\ &\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau} \end{aligned} \tag{12}$$

Using Eq. (12), the governing equations can be expressed as:

$$\begin{aligned} &\nabla \cdot \mathbf{v} = 0 \\ &\rho \frac{D\mathbf{v}}{Dt} = -\boldsymbol{\nabla} \mathscr{P} + \nabla \cdot \boldsymbol{\tau} \end{aligned} \tag{13}$$

For creeping flow, the inertia effect is negligible, so Eq. (13) is simplified as follows:

$$\begin{aligned} &\nabla \cdot \mathbf{v} = 0 \\ &\boldsymbol{\nabla} \mathscr{P} = \nabla \cdot \boldsymbol{\tau} \end{aligned} \tag{14}$$

The Creeping Flow and Modified Pressure

The governing equations of creeping flow of the second order fluid can be obtained by substituting Eq. (8) into the Eq. (14):

$$(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = 0 \tag{15}$$

$$[\boldsymbol{\nabla} \cdot \{b_1 \boldsymbol{\gamma}_{(1)} + b_2 \boldsymbol{\gamma}_{(2)} + b_{11} \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)}\}] = \boldsymbol{\nabla} \mathscr{P}$$

For a Newtonian flow with identical viscosity of the second order fluid flow, we have:

$$[\boldsymbol{\nabla} \cdot b_1 \boldsymbol{\gamma}_{(1)}] = b_1 \boldsymbol{\nabla}^2 \boldsymbol{v} = \boldsymbol{\nabla} \mathscr{P}_N \tag{16}$$

where $\mathscr{P}_N$ is the modified pressure of Newtonian flow.

The *Giesekus equation* is usually used to proof some useful theorems about the flow of second order fluid:

$$[\boldsymbol{\nabla} \cdot \{\boldsymbol{\gamma}_{(2)} + \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)}\}] = \boldsymbol{\nabla}\left[\frac{1}{b_1} \frac{D}{Dt} \mathscr{P}_N + \frac{1}{4}(\boldsymbol{\gamma}_{(1)} : \boldsymbol{\gamma}_{(1)})\right] \tag{17}$$

Some Useful Theorems

(a) The Three-Dimensional Flow Theorem of Giesekus

Given a velocity field $\boldsymbol{v}$ and a pressure field $\mathscr{P}_N$ that satisfy the equations for creeping flow of an incompressible Newtonian fluid, then the same velocity field $\boldsymbol{v}$ and the pressure field $\mathscr{P}$ given by Eq. (18) satisfy the equations for creeping flow of an incompressible second-order fluid with $b_{11} = b_2$.

$$\mathscr{P} = \mathscr{P}_N + \frac{b_2}{b_1}\frac{D}{Dt}\mathscr{P}_N + \frac{b_2}{4}(\boldsymbol{\gamma}_{(1)}:\boldsymbol{\gamma}_{(1)}) \tag{18}$$

Based on (9), the mentioned constraint in above theorem means:

$$b_{11} = b_2 \quad \longrightarrow \quad -\Psi_{2,0} / \Psi_{1,0} = \frac{1}{2} \tag{19}$$

The experimental values of this ratio are generally 1/4 or smaller. This theorem is mostly useful to check the validity of CFD simulations.

(b) The Plane Flow Theorem of Tanner and Pipkin

Given a plane velocity field $\boldsymbol{v}$ and a pressure field $\mathscr{P}_N$ that satisfy the equations for creeping plane flow of an incompressible Newtonian fluid, then the same velocity field and the pressure field $\mathscr{P}$ given by Eq. (20) satisfy the equations for creeping plane flow of an incompressible second-order fluid.

$$\mathscr{P} = \mathscr{P}_N + \frac{b_2}{b_1}\frac{D}{Dt}\mathscr{P}_N + \left(\frac{b_{11}}{2} - \frac{b_2}{4}\right)(\boldsymbol{\gamma}_{(1)} : \boldsymbol{\gamma}_{(1)}) \tag{20}$$

This theorem is limited to plane flow defined by

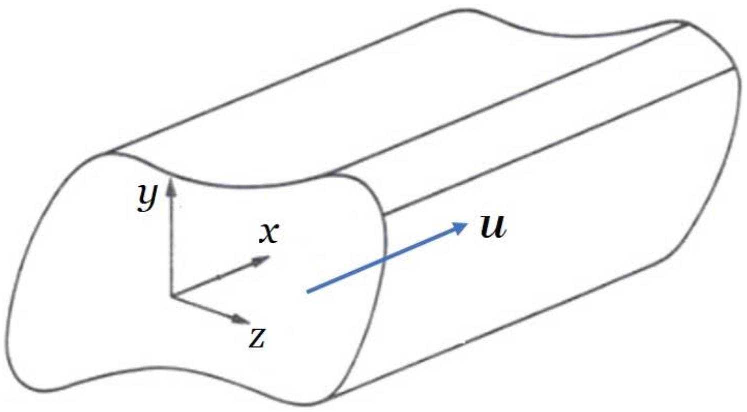
$$u = u(x, y), \quad v = v(x, y) \quad \& \quad w = 0 \tag{21}$$

It is important to mention that this theorem has no any constraint for fluid properties so it is so more useful than the previous theorem.

(c) The Rectilinear Flow Theorem of Langlois, Rivlin, and Pipkin

Given a rectilinear velocity field $\boldsymbol{v}$ and a pressure field $\mathscr{P}_N$ that satisfy the equation of motion for a Newtonian fluid and such that $\nabla\mathscr{P}_N$ does not change with distance in the flow direction, then the same velocity field and the pressure field $\mathscr{P}$ given by Eq. (22) satisfy the equations of continuity and motion for an incompressible second-order fluid.

$$\mathscr{P} = \mathscr{P}_N + \frac{b_2}{b_1}\frac{\partial}{\partial t}\mathscr{P}_N + \frac{b_{11}}{b_1}(\boldsymbol{v}\cdot\nabla\mathscr{P}_N) + \frac{b_2}{4}(\dot{\boldsymbol{\gamma}}:\dot{\boldsymbol{\gamma}}) + \frac{(b_{11}-b_2)}{2}((\nabla\boldsymbol{v}):(\nabla\boldsymbol{v})^{\dagger}) \tag{22}$$



This theorem is limited to rectilinear flow defined by

$$u = u(y,z), \quad v = 0 \quad \& \quad w = 0 \tag{23}$$

The theorem is useful to analyze the fully developed flow of second order fluid in ducts with arbitrary shape of cross section.

Quasilinear Models

The quasilinear models are constitutive equations that are able to model the flow of viscoelastic liquids but they bring constant values for viscometric functions. They are usually used in analytical and numerical studies due to its simplicity. Among these models, using the Oldroyd-B model is more common because most of nonlinear differential constitutive equations are simplified to this equation and it is also useful to model the flow of Boger liquids (constant viscosity viscoelastic fluids).

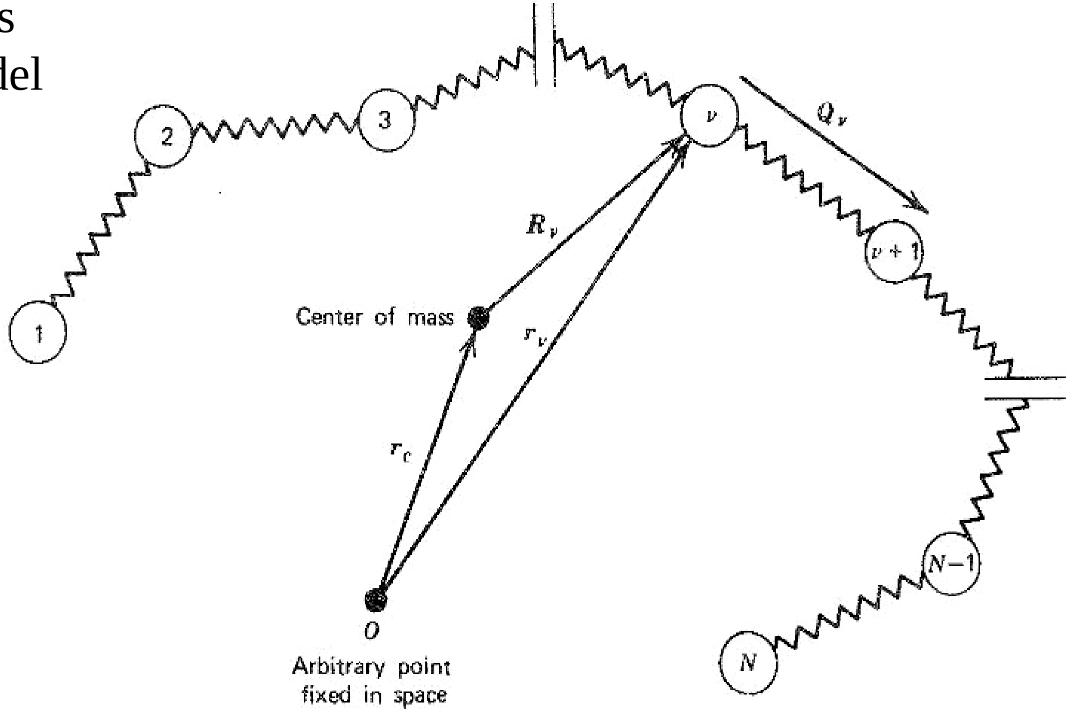
Viscometric functions of some quasilinear constitutive equations

Model	**Equation**	η	Ψ_1	Ψ_2
Upper Convected Maxwell (UCM)	$\boldsymbol{\tau} + \lambda \boldsymbol{\tau}_{(1)} = \eta_0 \boldsymbol{\gamma}_{(1)}$	η_0	$2\eta_0\lambda$	0
Oldroyd-A	$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau}^{(1)} = \eta_0 \left(\boldsymbol{\gamma}^{(1)} + \lambda_2 \boldsymbol{\gamma}^{(2)} \right)$	η_0	$2\eta_0 \left(\lambda_1 - \lambda_2 \right)$	$-\Psi_1$
Oldroyd-B	$\boldsymbol{\tau} + \lambda_1 \boldsymbol{\tau}_{(1)} = \eta_0 \left(\boldsymbol{\gamma}_{(1)} + \lambda_2 \boldsymbol{\gamma}_{(2)} \right)$	η_0	$2\eta_0 \left(\lambda_1 - \lambda_2 \right)$	0
Second Order Fluid (SOF)	$\boldsymbol{\tau} = \eta_0 \left(\boldsymbol{\gamma}_{(1)} + \lambda_1 \boldsymbol{\gamma}_{(2)} + \lambda_2 \left\{ \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)} \right\} \right)$	η_0	$-2\eta_0\lambda_1$	$\eta_0\lambda_2$

Nonlinear Models

Since 1950, a lot of nonlinear constitutive equations have been presented. They could model the large deformation with non-zero relaxation time, nonlinear viscometric functions and extensional viscosity.

1. Retarded-Motion Expansion Models: third order fluids and higher order models
2. Finitely-Extensible-Nonlinear-Elastic Dumbbell Model
3. Multi-constants Oldroyd models
4. Criminale-Ericksen-Filbey Model
5. Reiner-Rivlin Model
6. White-Metzner Model
7. Giesekus Model
8. Phan-Thien-Tanner Model
9. Bird-Carreau
10. Grin-Rivlin Model
11. Lodge-Maxwell Model
12. Doi-Edwards Model
13. Kaye-BKZ Model
14. Curtiss-Bird Model
15. Leonov Model



The freely jointed bead-spring chain model

The Criminale-Ericksen-Filbey (CEF) Model

The Criminale-Ericksen-Filbey (CEF) Constitutive Equation

The Criminale-Ericksen-Filbey (CEF) equation includes as a special case the "generalized Newtonian fluid" model, which has been widely used and is still being used for industrial calculations because of its simplicity. The equation of this model is:

$$\boldsymbol{\tau} = \eta(q)\boldsymbol{\gamma}_{(1)} - \frac{1}{2}\Psi_1(q)\boldsymbol{\gamma}_{(2)} + \Psi_2(q)\left\{\boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)}\right\}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (24)

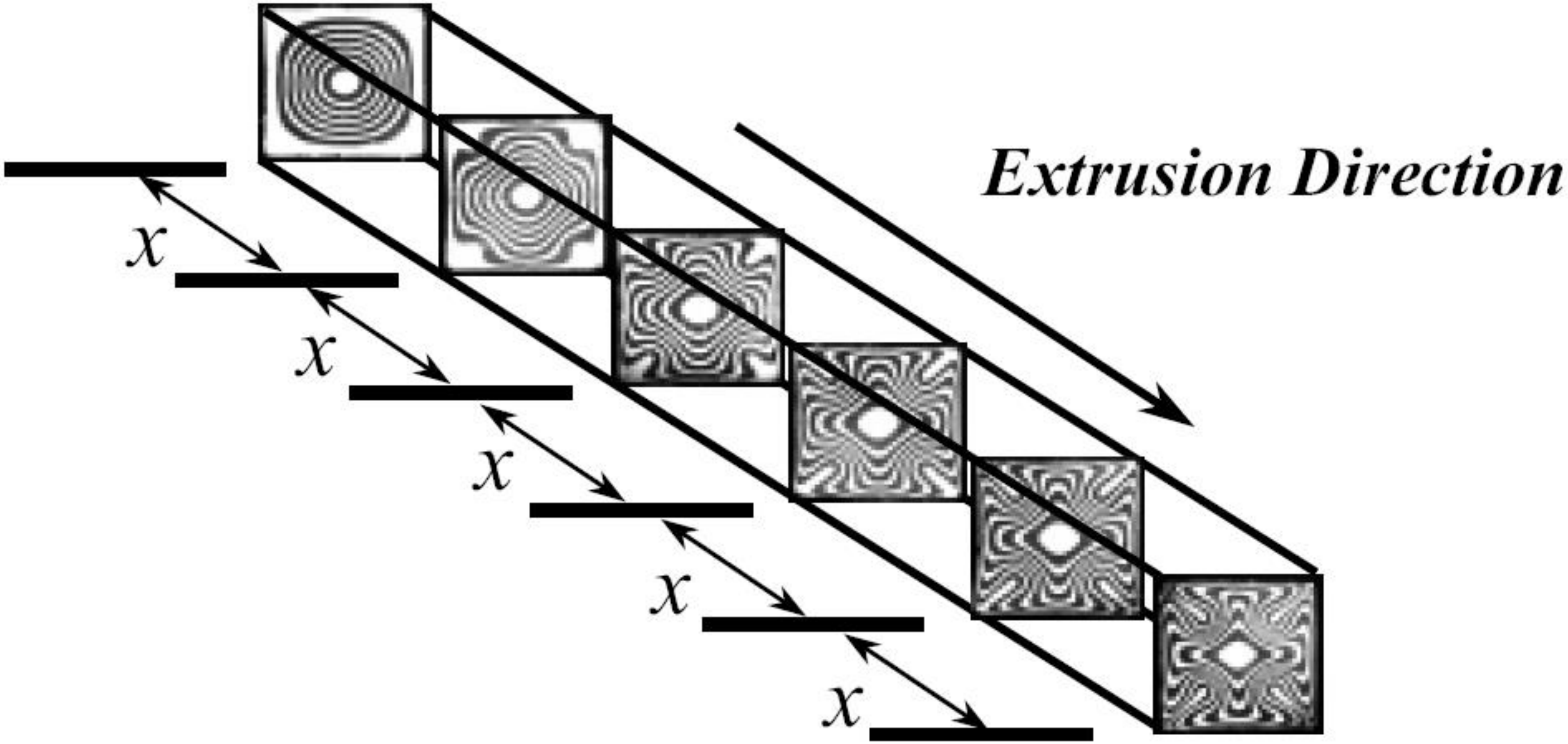
$$\boldsymbol{\tau} = \eta(q)\boldsymbol{\gamma}^{(1)} - \frac{1}{2}\Psi_1(q)\boldsymbol{\gamma}^{(2)} + \left(\Psi_1(q) + \Psi_2(q)\right)\left\{\boldsymbol{\gamma}^{(1)} \cdot \boldsymbol{\gamma}^{(1)}\right\}$$

where q is the generalized shear rate and $\eta(q)$, $\Psi_1(q)$ and $\Psi_2(q)$ are the viscometric functions and they are obtained using suitable equations such as power-law, Cross and so on. For steady-state unidirectional shear flows, the CEF equation is applicable and it is considered as the exact model. It is very helpful in the fluid mechanical analysis of standard rheometers. This model brings the second normal stress difference so it is a useful to model the phenomenon that affected by this material modulus. For example, it is possible to model the corner vortices in flow of viscoelastic fluids in straight ducts with noncircular cross section.

The model is not useful for unsteady flows because it does not present any relaxation time. Therefore, the model is restricted to the viscometric part of Pipkin diagram. The response of model for extensional flow is not reasonable.

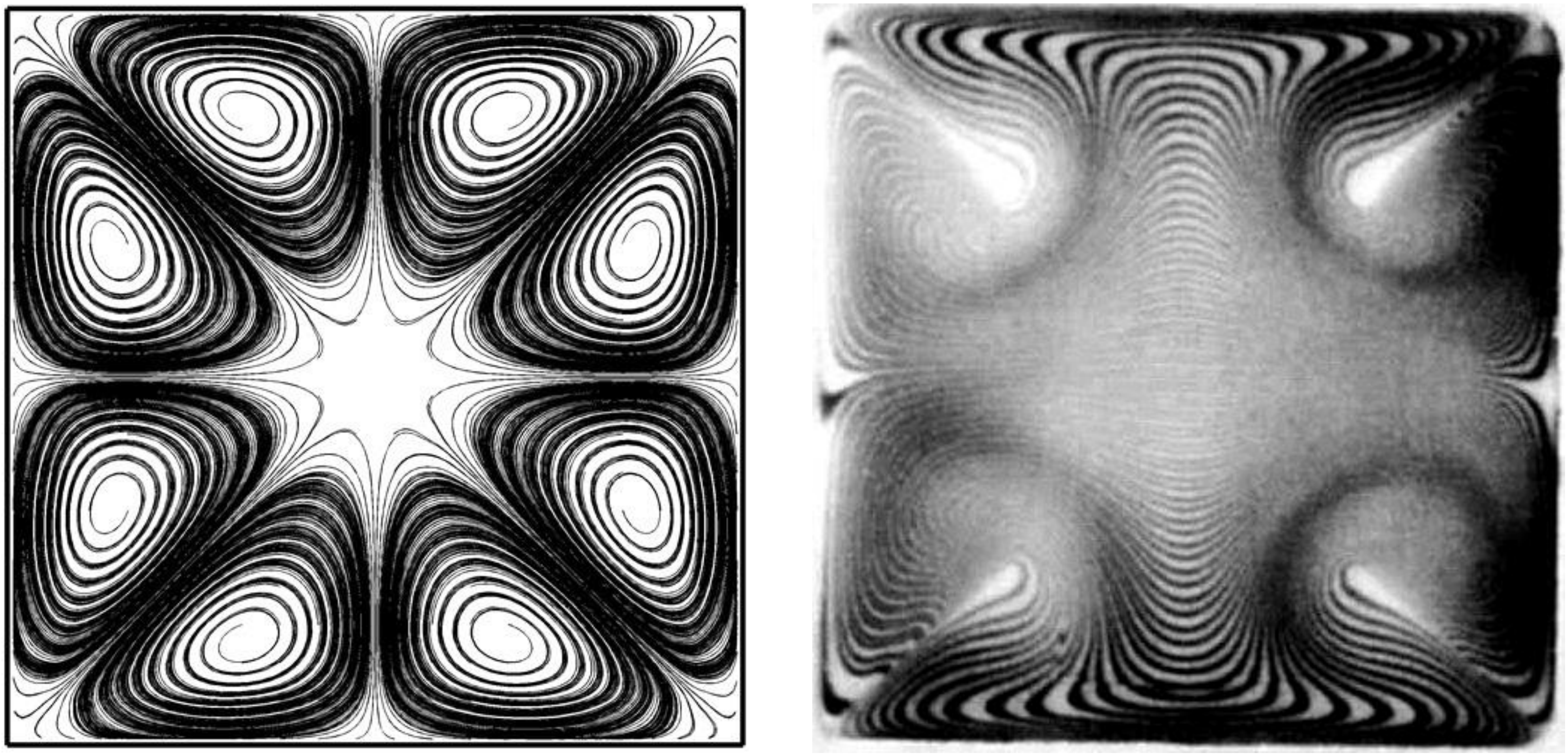
The model is reduced to the second order fluid by considering the constant viscometric functions. It also tends to the generalized Newtonian fluids for zero normal stress differences.

Flow inside non-Circular Straight Ducts



Structures of multi layer polystyrene in extrusion inside a straight duct.

Flow inside non-Circular Straight Ducts

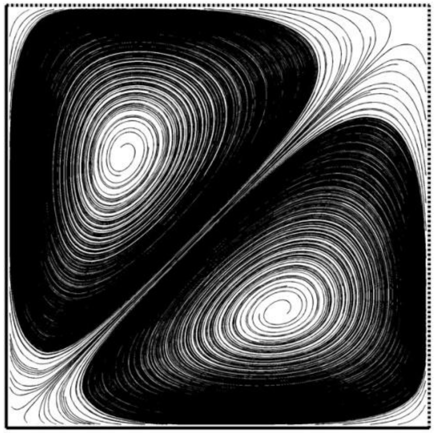


(a) **(b)**

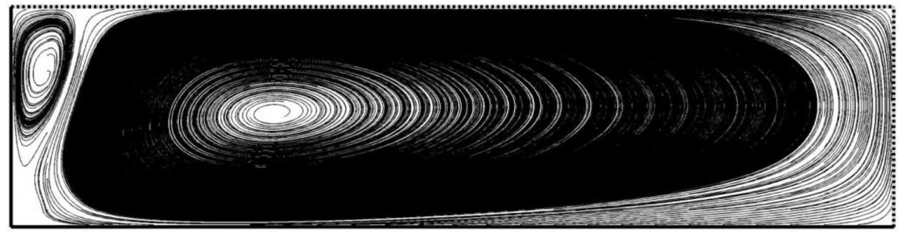
Viscoelastic flow in straight ducts, (a): secondary flows (numerical simulation), (b): material structure (experimental observation)

Flow inside non-Circular Straight Ducts

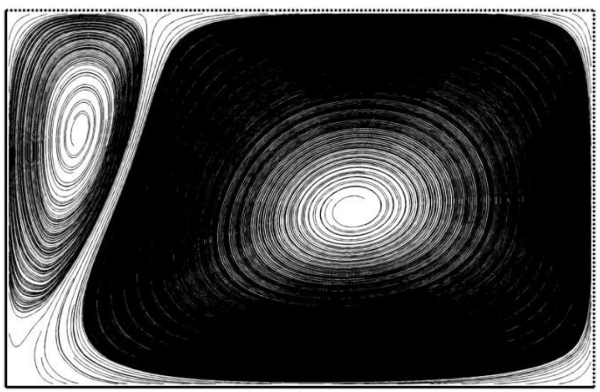
$\kappa = 1.0,\ S_{max} = 1.43\times10^{-3}$



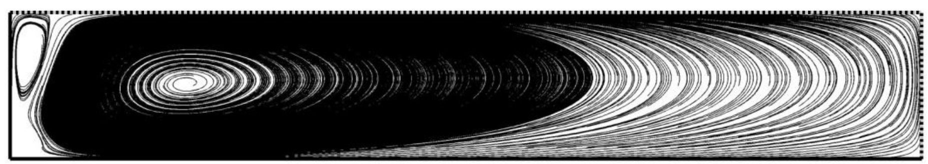
$\kappa = 4.0,\ S_{max} = 1.08\times10^{-4}$



$\kappa = 1.56,\ S_{max} = 9.55\times10^{-4}$



$\kappa = 6.25,\ S_{max} = 4.68\times10^{-5}$



Effect of aspect ratio on corner vortices (numerical simulation)

The Reiner-Rivlin Equation

The Reiner-Rivlin Equation

The Reiner-Rivlin constitutive equation is arisen from the retarded-motion expansion equation. This model is presented for steady homogenous irrotational viscoelastic flows. Remember that the following relation is existed for velocity gradient:

$$\nabla\mathbf{V}=\frac{1}{2}\left(\nabla\mathbf{V}+\nabla\mathbf{V}^{T}\right)+\frac{1}{2}\left(\nabla\mathbf{V}-\nabla\mathbf{V}^{T}\right)=\mathbf{D}+\mathbf{W} \qquad (25)$$

where **D** is deformation rate tensor and **W** is spin tensor:

$$\mathbf{D}=\frac{1}{2}\left(\nabla\mathbf{V}+\nabla\mathbf{V}^{T}\right) \quad \& \quad \mathbf{W}=\frac{1}{2}\left(\nabla\mathbf{V}-\nabla\mathbf{V}^{T}\right) \qquad (26)$$

W is an antisymmetric tensor and the absolute value of components of this tensor are same as the components of vorticity vector:

$$\left.\begin{array}{l}\mathbf{W}=\dfrac{1}{2}\begin{bmatrix}0 & \dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y} & \dfrac{\partial w}{\partial x}-\dfrac{\partial u}{\partial z}\\ \dfrac{\partial u}{\partial y}-\dfrac{\partial v}{\partial x} & 0 & \dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}\\ \dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x} & \dfrac{\partial v}{\partial z}-\dfrac{\partial w}{\partial y} & 0\end{bmatrix}\\ \boldsymbol{\omega}=\nabla\times\mathbf{V}=\left(\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}\right)\mathbf{i}+\left(\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}\right)\mathbf{j}+\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)\mathbf{k}\end{array}\right\}\rightarrow\mathbf{W}=\begin{bmatrix}0 & \omega_z & -\omega_y\\ -\omega_z & 0 & \omega_x\\ \omega_y & -\omega_x & 0\end{bmatrix} \qquad (27)$$

The Reiner-Rivlin Equation

From Eqns. (25) and (27), it can be concluded that for irrotational flow: $\mathbf{W}=0$ and $\nabla\mathbf{V}=\nabla\mathbf{V}^T$.
Therefore, from the definition of shear rate tensor, we have:

$$\dot{\gamma}=\gamma_{(1)}=\nabla\mathbf{V}+\nabla\mathbf{V}^T=2\nabla\mathbf{V} \quad\longrightarrow\quad \nabla\mathbf{V}=\nabla\mathbf{V}^T=\frac{1}{2}\gamma_{(1)} \qquad (28)$$

In steady homogenous flows, the rate of changes and advections are ignored:

$$\frac{D\gamma_{(n)}}{Dt}=\frac{\partial\gamma_{(n)}}{\partial t}+\mathbf{V}.\nabla\gamma_{(n)}=0 \qquad (29)$$

Therefore, the n+1 order of contravariant convected derivative of shear rate tensor is simplified as:

$$\gamma_{(n+1)}=\frac{D\gamma_{(n)}}{Dt}-\left\{\nabla\mathbf{V}^T\gamma_{(n)}+\gamma_{(n)}\nabla\mathbf{V}\right\}=-\left\{\nabla\mathbf{V}^T\gamma_{(n)}+\gamma_{(n)}\nabla\mathbf{V}\right\}=-\frac{1}{2}\left\{\gamma_{(1)}\gamma_{(n)}+\gamma_{(n)}\gamma_{(1)}\right\} \qquad (30)$$

From Eq. (30), we can find the following relationship:

$$\gamma_{(2)}=-\frac{1}{2}\left\{\gamma_{(1)}\gamma_{(1)}+\gamma_{(1)}\gamma_{(1)}\right\}=-\gamma_{(1)}{}^2$$

$$\gamma_{(3)}=-\frac{1}{2}\left\{\gamma_{(1)}\gamma_{(2)}+\gamma_{(2)}\gamma_{(1)}\right\}=-\frac{1}{2}\left\{-\gamma_{(1)}\gamma_{(1)}{}^2-\gamma_{(1)}{}^2\gamma_{(1)}\right\}=+\gamma_{(1)}{}^3 \qquad (31)$$

$$\vdots$$

$$\gamma_{(n)}=(-1)^{n+1}\gamma_{(1)}{}^n$$

The Reiner-Rivlin Equation

From the Cayley-Hamilton theorem, the following relationship is existed for a tensor like **A**:

$$\mathbf{A}^3 - I_1\mathbf{A}^2 + I_2\mathbf{A} - I_3\boldsymbol{\delta} = 0 \qquad (32)$$

where I_1, I_2 and I_3 are the eigenvalues of tensor **A** and $\boldsymbol{\delta}$ is identity tensor. From this theorem, it is possible to derive the following relation for shear rate tensor:

$$\boldsymbol{\gamma}_{(1)}^{\;3} - I_1\boldsymbol{\gamma}_{(1)}^{\;2} + I_2\boldsymbol{\gamma}_{(1)} - I_3\boldsymbol{\delta} = 0 \quad \rightarrow \quad \boldsymbol{\gamma}_{(1)}^{\;3} = I_1\boldsymbol{\gamma}_{(1)}^{\;2} - I_2\boldsymbol{\gamma}_{(1)} + I_3\boldsymbol{\delta} = g_3(\boldsymbol{\gamma}_{(1)}, \boldsymbol{\gamma}_{(1)}^{\;2}) \qquad (33)$$

From Eq. (33), we know that we can derive the third power of shear rate tensor $(\boldsymbol{\gamma}_{(1)}^{\;3})$ in terms of $\boldsymbol{\gamma}_{(1)}$ and $\boldsymbol{\gamma}_{(1)}^{\;2}$. For the higher order powers, we have:

$$\boldsymbol{\gamma}_{(1)}^{\;4} = \boldsymbol{\gamma}_{(1)} \cdot \boldsymbol{\gamma}_{(1)}^{\;3} = \boldsymbol{\gamma}_{(1)} g_3(\boldsymbol{\gamma}_{(1)}, \boldsymbol{\gamma}_{(1)}^{\;2}) = h(\boldsymbol{\gamma}_{(1)}, \boldsymbol{\gamma}_{(1)}^{\;2}, \boldsymbol{\gamma}_{(1)}^{\;3}) \xrightarrow{\text{From Theorem}} \boldsymbol{\gamma}_{(1)}^{\;4} = g_4(\boldsymbol{\gamma}_{(1)}, \boldsymbol{\gamma}_{(1)}^{\;2})$$

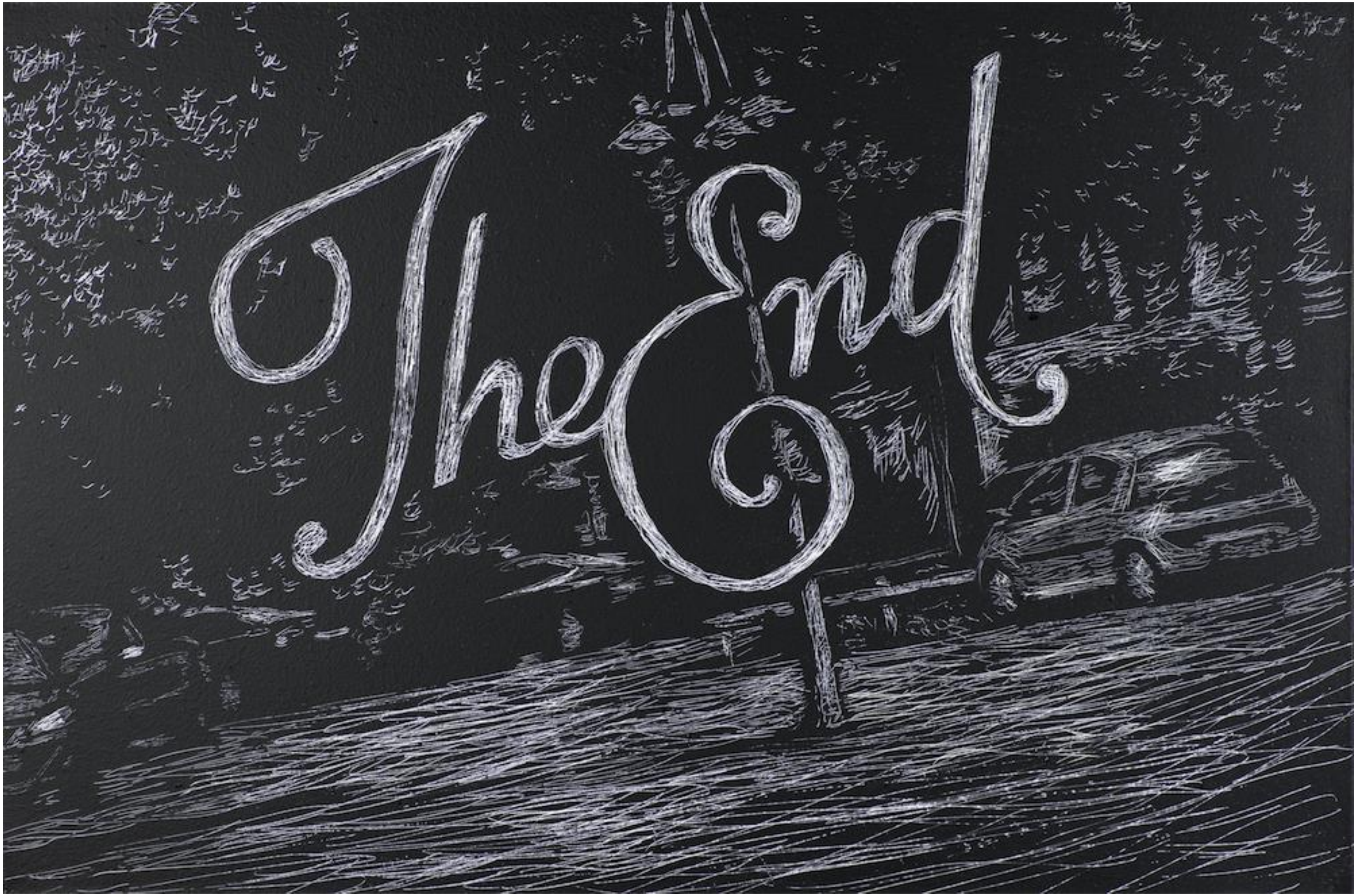
$$\vdots \qquad (34)$$

$$\boldsymbol{\gamma}_{(1)}^{\;n} = g_n(\boldsymbol{\gamma}_{(1)}, \boldsymbol{\gamma}_{(1)}^{\;2})$$

Therefore, the retarded-motion expansion equation collapses to:

$$\boldsymbol{\tau} = f_1(II, III)\dot{\boldsymbol{\gamma}} + f_2(II, III)\{\dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}}\} \qquad (35)$$

This is the Reiner-Rivlin equation. When originally proposed this equation was believed to be more generally applicable than it is. Hence there are unfortunately many publications in which the Reiner-Rivlin equation has been applied inappropriately to flows that are not steady, homogeneous, and irrotational.


The End