Non-Newtonian Fluid Mechanics

(Part - VIII)

Mechanical Engineering Department,

Shahrood University of Technology

By M. Norouzi

May 2020

Some Points on Linear Viscoelastic Models

By parallel and series connections of different damper and spring elements, we can derive the different linear constitutive equations. It is possible to show that the general form of these equations is:

$$\tau + \alpha_1 \dot{\tau} + \alpha_2 \ddot{\tau} + \alpha_3 \dddot{\tau} + \ldots + \alpha_n \frac{d^n \tau}{dt^n} = \eta \left(\dot{\gamma} + \beta_1 \ddot{\gamma} + \beta_2 \dddot{\gamma} + \ldots + \beta_m \frac{d^{m+1} \gamma}{dt^{m+1}} \right) \qquad (1)$$

The above linear equation brings a constant viscosity and, n and m discrete relaxation and retardation times, respectively. They are appropriate for small deformations while their results are not usually correct for large deformations and fluid flow. These models are widely used in solid mechanics and polymer Engineering for the problems in which the size of deformations is small. They are also useful for interoperating the results of some standard rheological tests such as stress relaxation, creep, recoil, frequency sweep and amplitude sweep tests. The linear equations cannot model the nonlinear viscometric functions (viscosity and, the first and second normal stress differences coefficients) and extensional viscosity. Therefore, there are not suitable to model the flow of viscoelastic fluids. Some nonlinear and quasilinear constitutive equations for flow of viscoelastic liquids are derived base on generalization of linear models.

Modeling (Non-Linear Viscoelasticity)

Linear viscoelastic constitutive equations are not suitable for modeling the large deformations of viscoelastic materials. There are not also able to present the non-linar viscometric functions in shear flows and elongational viscosity.

Oldroyd's Principles

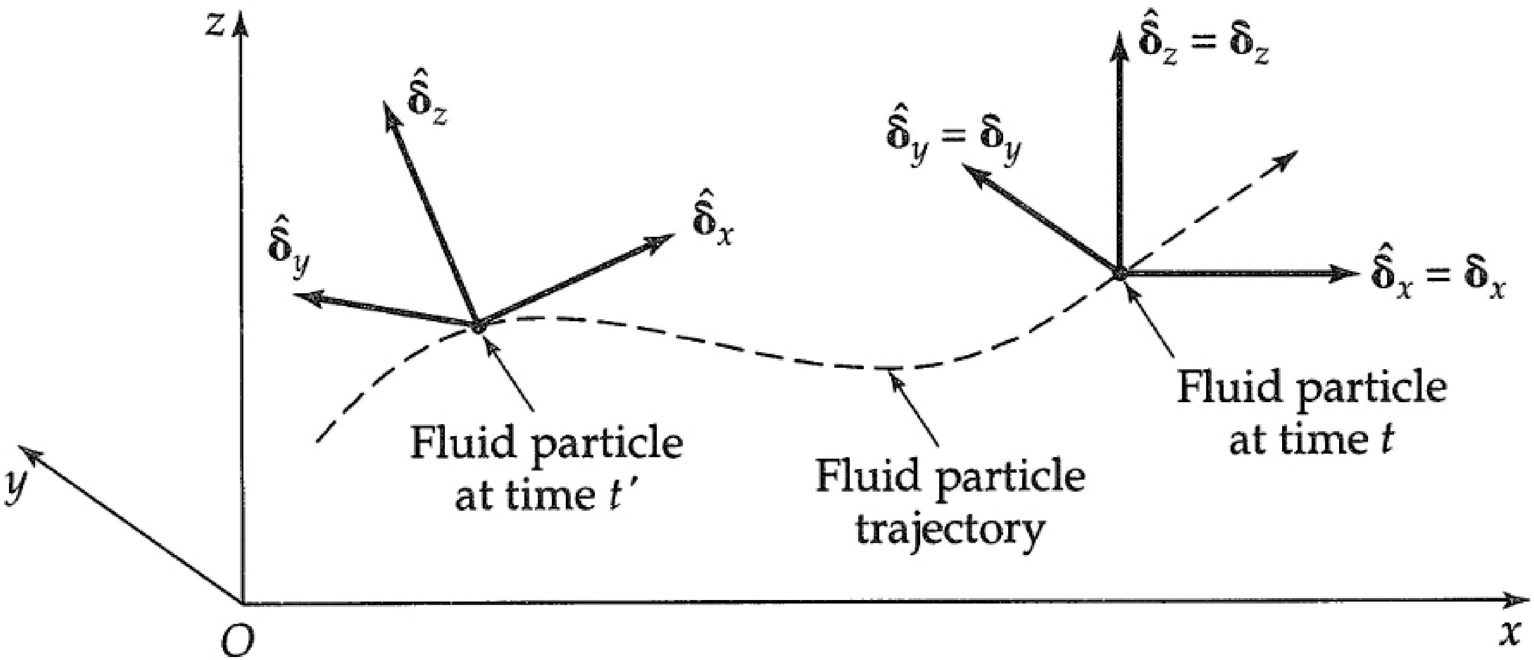
The continuum principles of non-linear constitutive equations have been presented by Oldroyd (1950 and 1964). He proposed that the constitutive equations should be invariant from

(a) Change and orthogonal transformation of coordinate systems
(b) Translation and rigid body motion of fluid elements
(c) Change of rheological history of neighboring fluid elements

Now, we introduce the hypothesis, that the relation between the stress tensor and the kinematic tensors at a fluid particle should be independent of the instantaneous orientation of that particle in space. This seems like a reasonable hypothesis; if you measure the stress-strain relation in a rubber band, it should not matter whether you are stretching the rubber band in the north-south direction or the east-west direction, or even rotating as you take data (provided, of course, that you do not rotate so rapidly that centrifugal forces interfere with the measurements).

Convected Coordinate System

One way to implement the mentioned hypothesis is to introduce at each fluid particle a corotating coordinate frame. This orthogonal frame rotates with the local instantaneous angular velocity as it moves along with the fluid particle through space. In the corotating coordinate system we can now write down some kind of relation between the stress tensor and the rate-of-strain tensor.



The convected coordinate system which is moved with fluid particles

Convected Coordinate System

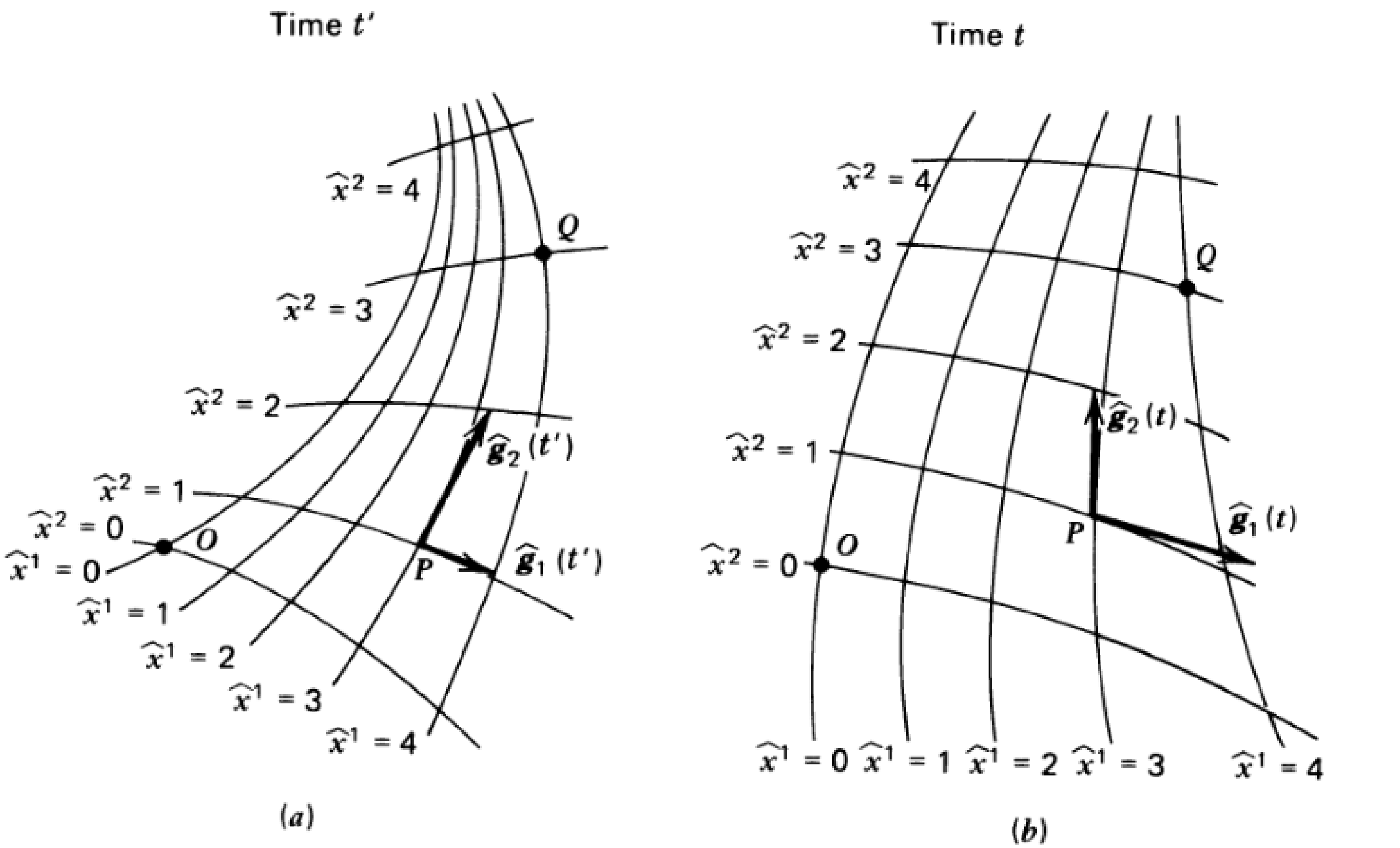
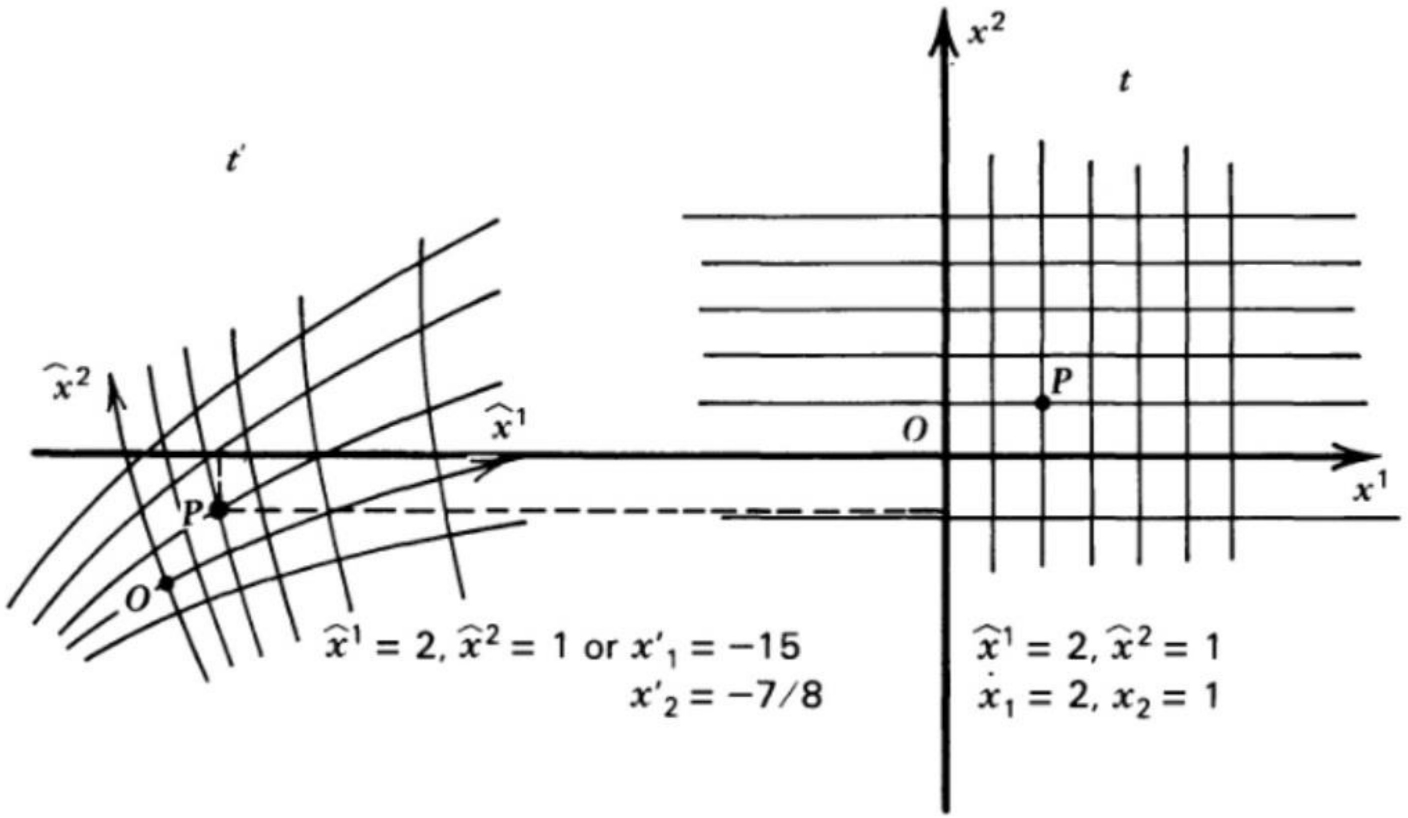


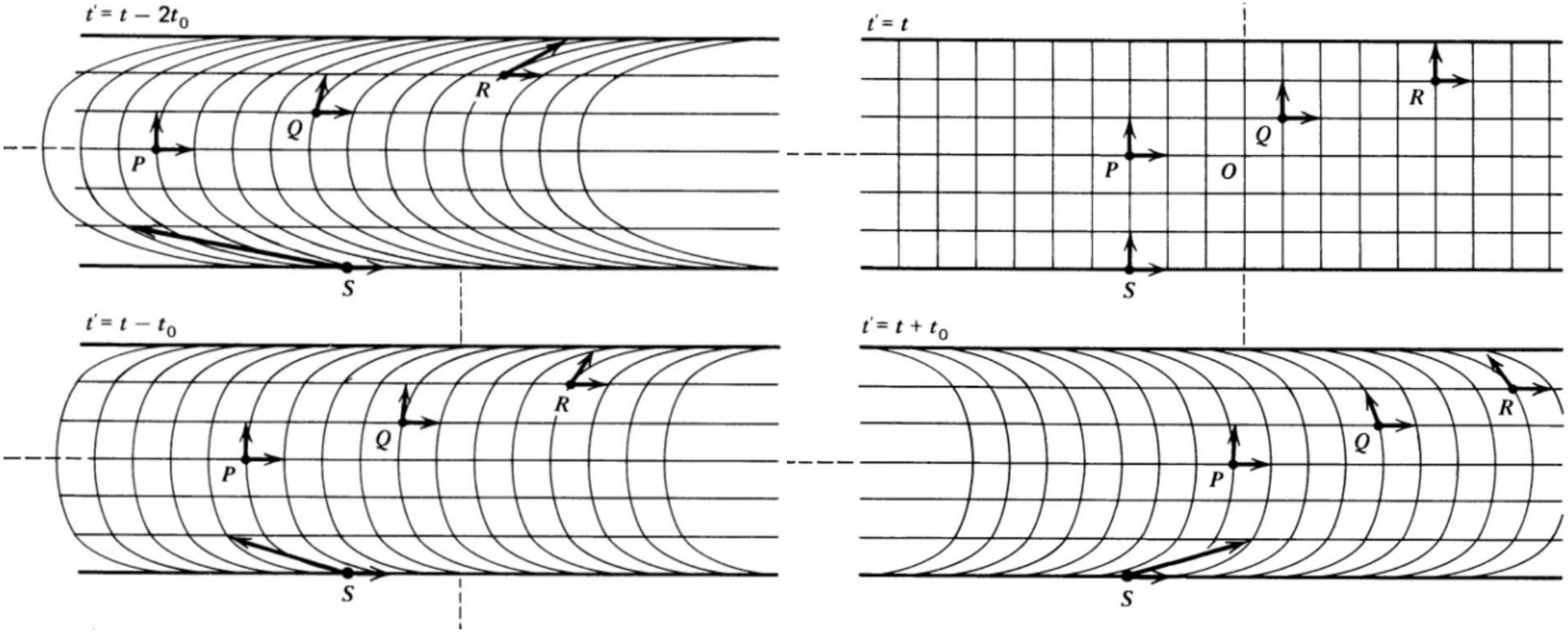
FIGURE 9.1-1. An arbitrarily chosen coordinate system, embedded in a flowing fluid, at two different times (a) t' and (b) t. Fluid particle P is located at $\hat{x}^1 = 3$, $\hat{x}^2 = 1$ at all times; fluid particle Q is at $\hat{x}^1 = 4$, $\hat{x}^2 = 3$ at all times. The base vectors $\hat{\boldsymbol{g}}_1$ and $\hat{\boldsymbol{g}}_2$ at fluid particle P are also shown.

Convected Coordinate System



Convected coordinates; the convected coordinates $\hat{x}^i$ exactly coincide with the fixed coordinates x_i at time t.

Convected Coordinate System for a Flow



Steady flow between two parallel plates. The fluid viscosity is given by $\eta = m\dot{\gamma}^{n-1}$ with $n = \frac{1}{3}$. At $t' = t$ fluid particle R has coordinates $x_1 = 5, x_2 = 2, x_3 = 0$ with respect to the origin O of a Cartesian coordinate system. At time $t' = t - 2t_0$ its coordinates are $x'_1 = 1, x'_2 = 2, x'_3 = 0$ with respect to the origin O. Its convected coordinates are $\hat{x}^1 = 5, \hat{x}^2 = 2, \hat{x}^3 = 0$ for all times t'. The convected base vectors $\hat{\boldsymbol{g}}_1$ and $\hat{\boldsymbol{g}}_2$ are shown for fluid particles P, Q, R, S; $\hat{\boldsymbol{g}}_3$ is of unit length perpendicular to the plane of the paper. At $t' = t$ the convected base vectors $\hat{\boldsymbol{g}}_1, \hat{\boldsymbol{g}}_2, \hat{\boldsymbol{g}}_3$ coincide with the unit vectors $\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \boldsymbol{\delta}_3$ of the Cartesian coordinate system.

Convected Coordinate System

We first examine the kinematic description of the flow. At any point in the convected coordinate system we can construct a set of three *convected base vectors* $\hat{\boldsymbol{g}}_i = (\partial/\partial\hat{x}^i)\boldsymbol{r}$, where $\boldsymbol{r}$ is the position vector (see §A.8). A base vector $\hat{\boldsymbol{g}}_i$ is tangent to the $\hat{x}^i$-coordinate curve, and its change in length indicates the extent to which the fluid is stretched in the $\hat{x}^i$-direction. The base vectors depend on the convected coordinates $\hat{x} \equiv \hat{x}^1, \hat{x}^2, \hat{x}^3$ and on the time. The vector between two neighboring fluid particles $\hat{x}$ and $\hat{x} + d\hat{x}$ at time t' is given by

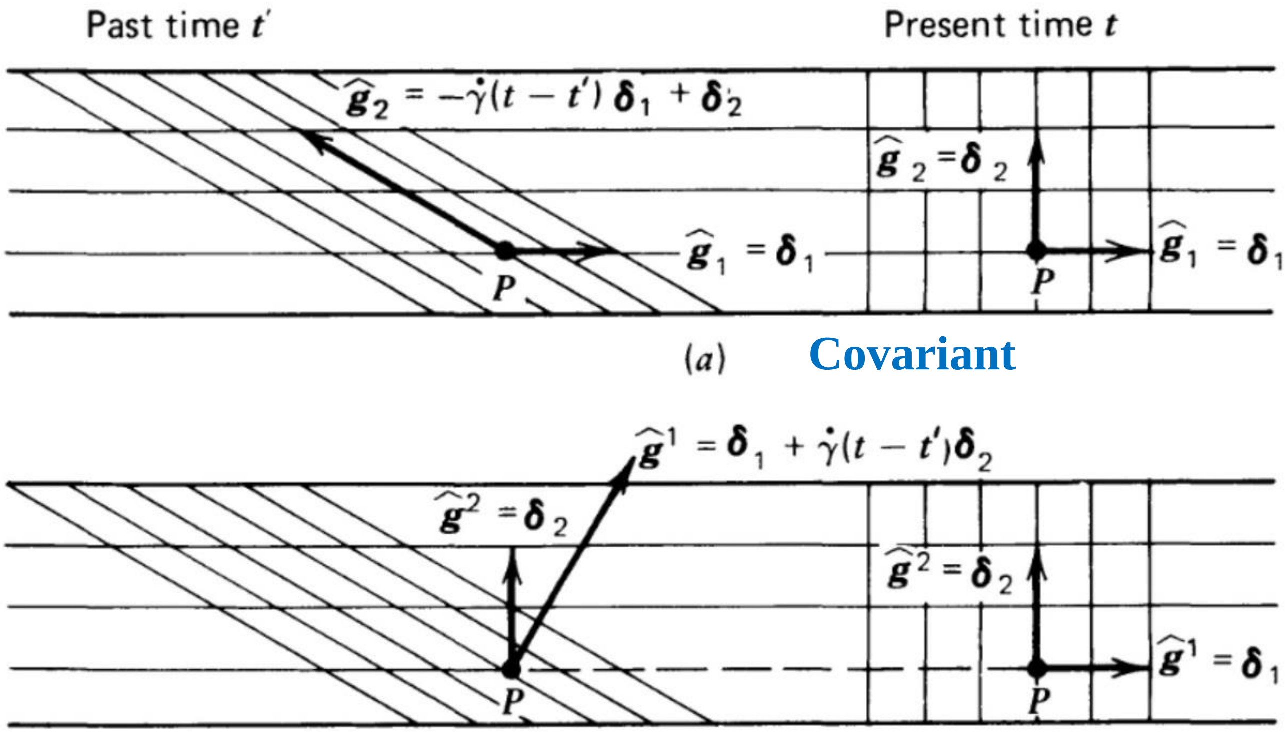
$$d\boldsymbol{r} = \sum_i \hat{\boldsymbol{g}}_i(\hat{x}, t')d\hat{x}^i \tag{2}$$

so that the square of the separation between the particles is given by

$$\begin{aligned}(d\boldsymbol{r} \cdot d\boldsymbol{r}) &= \sum_i \sum_j (\hat{\boldsymbol{g}}_i \cdot \hat{\boldsymbol{g}}_j) d\hat{x}^i\, d\hat{x}^j \\ &= \sum_i \sum_j \hat{g}_{ij}(\hat{x}, t')d\hat{x}^i\, d\hat{x}^j\end{aligned} \tag{3}$$

in which the $\hat{g}_{ij}(\hat{x}, t)$ are the covariant metric coefficients. These quantities, which describe the relative distances between arbitrary pairs of neighboring particles, contain complete information about the shape of a fluid element at time t'. All the kinematic quantities that may appear in a constitutive equation must be derivable from the metric coefficients $\hat{g}_{ij}(\hat{x}, t)$ for $-\infty < t' \leq t$, which is sometimes called the *deformation history* at the fluid particle $\hat{x}$. Since the $\hat{g}_{ij}$ are obtained by forming scalar products, they do not depend on the instantaneous position or orientation of the fluid element in space.

Types of Convected Coordinate Systems



(a) **Covariant**

(b) **Contravariant**

Steady shear flow $v_1 = \dot{\gamma}x_2$, $v_2 = 0$, $v_3 = 0$ showing the convected base vectors and reciprocal base vectors associated with a fluid particle P as it moves along from some past time t' to the present time t. The vectors $\hat{\boldsymbol{g}}_3$ and $\hat{\boldsymbol{g}}^3$ are perpendicular to the plane of the paper and both equal to $\boldsymbol{\delta}_3$. In this *homogeneous* flow the convected base vectors $\hat{\boldsymbol{g}}_i$ are coincident with material lines; that is, they are "embedded" in the fluid. The convected reciprocal base vectors $\hat{\boldsymbol{g}}^i$ are perpendicular to material surfaces.

Types of Convected Coordinate Systems

Hence we indicate the dependence on fluid particle and time by writing $\hat{\boldsymbol{g}}_i(\hat{x}^1, \hat{x}^2, \hat{x}^3, t')$ or alternatively $\hat{\boldsymbol{g}}_i(\boldsymbol{r}, t, t')$—that is, we can use the particle label $\hat{x}^i (i = 1, 2, 3)$ or $(\boldsymbol{r}, t)$. Then,

$$\hat{\boldsymbol{g}}_i(\boldsymbol{r}, t, t') = \frac{\partial}{\partial \hat{x}^i} \boldsymbol{r}' \tag{4}$$

At every fluid particle we may also define a set of *convected reciprocal base vectors*

$$\hat{\boldsymbol{g}}^i(\boldsymbol{r}, t, t') = \frac{\partial}{\partial \boldsymbol{r}'} \hat{x}^i \tag{5}$$

It is possible to show that the time derivatives of the convected base vectors are

$$\boxed{\begin{aligned} \frac{\partial}{\partial t'} \hat{\boldsymbol{g}}_i &= + [\hat{\boldsymbol{g}}_i \cdot \nabla \boldsymbol{v}] \\ \frac{\partial}{\partial t'} \hat{\boldsymbol{g}}^i &= - [(\nabla \boldsymbol{v}) \cdot \hat{\boldsymbol{g}}^i] \end{aligned}} \tag{6}$$

Strain in Convected Coordinate Systems

Previously, in the discussion of linear viscoelasticity, we introduced infinitesimal strain tensors describing the strain at time t', relative to the fluid configuration at time t. This suggests that we ought to use the quantity $[\hat{g}_{ij}(\mathbf{r}, t, t') - \hat{g}_{ij}(\mathbf{r}, t, t)]$ as a measure of the relative strain, where $\mathbf{r}$, t is the label for a particular fluid particle.

$$[\hat{g}_{ij}(\mathbf{r}, t, t') - \hat{g}_{ij}(\mathbf{r}, t, t)] = \gamma_{ij}^{[0]}(\mathbf{r}, t, t') \tag{7}$$

$$-[\hat{g}^{ij}(\mathbf{r}, t, t') - \hat{g}^{ij}(\mathbf{r}, t, t)] = \gamma_{[0]ij}(\mathbf{r}, t, t') \tag{8}$$

Several comments deserve to be made regarding the relative strain tensors:

a. They are both obtainable from the displacement functions.

b. They both reduce to the infinitesimal strain tensor $\boldsymbol{\gamma}$ for small strains.

c. They both depend on the particle label $\mathbf{r}$, t and on the time t'; when $t' = t$, the relative finite strain tensors vanish.

d. They both arise naturally in molecular theories, and both are used in empirical constitutive equations.

Derivatives of Strain in Convected Coordinate Systems

In linear viscoelasticity we used not only the infinitesimal strain tensor $\boldsymbol{\gamma}$ but also the rate of strain tensor $\dot{\boldsymbol{\gamma}} = \partial\boldsymbol{\gamma}/\partial t$ and higher time derivatives, $\ddot{\boldsymbol{\gamma}} = \partial\dot{\boldsymbol{\gamma}}/\partial t = \partial^2\boldsymbol{\gamma}/\partial t^2$, etc. It is not surprising, then, that in nonlinear viscoelasticity time derivatives of the relative strain tensors have been defined

$$\boldsymbol{\gamma}^{[n]} = \partial^n \boldsymbol{\gamma}^{[0]}(\boldsymbol{r}, t, t')/\partial t'^n \quad \text{and} \quad \boldsymbol{\gamma}_{[n]} = \partial^n \boldsymbol{\gamma}_{[0]}(\boldsymbol{r}, t, t')/\partial t'^n \qquad (9)$$

Covariant Convected Derivative of Strain (Lower Convected Derivative)

$$\begin{aligned}
\boldsymbol{\gamma}^{[1]}(\boldsymbol{r}, t, t) &= \boldsymbol{\gamma}^{(1)}(\boldsymbol{r}, t) = \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\dagger \\
\boldsymbol{\gamma}^{[2]}(\boldsymbol{r}, t, t) &= \boldsymbol{\gamma}^{(2)}(\boldsymbol{r}, t) = \frac{D\boldsymbol{\gamma}^{(1)}}{Dt} + \{(\nabla\boldsymbol{v})\cdot\boldsymbol{\gamma}^{(1)} + \boldsymbol{\gamma}^{(1)}\cdot(\nabla\boldsymbol{v})^\dagger\} \\
\vdots \qquad & \qquad \vdots \qquad \qquad \qquad \vdots \\
\boldsymbol{\gamma}^{[n]}(\boldsymbol{r}, t, t) &= \boldsymbol{\gamma}^{(n)}(\boldsymbol{r}, t) = \frac{D\boldsymbol{\gamma}^{(n-1)}}{Dt} + \{\nabla\boldsymbol{v})\cdot\boldsymbol{\gamma}^{(n-1)} + \boldsymbol{\gamma}^{(n-1)}\cdot(\nabla\boldsymbol{v})^\dagger\}
\end{aligned} \qquad (10)$$

Derivatives of Strain in Convected Coordinate Systems

Contravariant Convected Derivative of Tensors (Upper Convected Derivative)

$$
\begin{aligned}
\boldsymbol{\gamma}_{[1]}(\boldsymbol{r}, t, t) &= \boldsymbol{\gamma}_{(1)}(\boldsymbol{r}, t) = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\dagger} \\
\boldsymbol{\gamma}_{[2]}(\boldsymbol{r}, t, t) &= \boldsymbol{\gamma}_{(2)}(\boldsymbol{r}, t) = \frac{D\boldsymbol{\gamma}_{(1)}}{Dt} - \{(\nabla \boldsymbol{v})^{\dagger} \cdot \boldsymbol{\gamma}_{(1)} + \boldsymbol{\gamma}_{(1)} \cdot \nabla \boldsymbol{v}\} \\
\vdots \qquad & \qquad \vdots \qquad\qquad\qquad \vdots \\
\boldsymbol{\gamma}_{[n]}(\boldsymbol{r}, t, t) &= \boldsymbol{\gamma}_{(n)}(\boldsymbol{r}, t) = \frac{D\boldsymbol{\gamma}_{(n-1)}}{Dt} - \{(\nabla \boldsymbol{v})^{\dagger} \cdot \boldsymbol{\gamma}_{(n-1)} + \boldsymbol{\gamma}_{(n-1)} \cdot \nabla \boldsymbol{v}\}
\end{aligned}
\tag{11}
$$

Some Tips about Convected Derivatives

a. The superscript [ ] and ( ) quantities arise from the behavior of the dot products of the convected base vectors $\hat{\boldsymbol{g}}_i$ whereas the subscript [ ] and ( ) quantities result from the dot products of the convected reciprocal base vectors $\hat{\boldsymbol{g}}^i$. The subscript quantities seem to arise more naturally in the molecular theories and have been used in most of the empirical constitutive equations.

b. The [ ] quantities, which depend on t and t', appear in integral constitutive equations, in integrals over t'. The ( ) quantities appear as functions of t in differential constitutive equations.

c. The [ ] quantities are identical to the corresponding ( ) quantities when $t' = t$; there are, however, no tensors $\boldsymbol{\gamma}^{(0)}$ and $\boldsymbol{\gamma}_{(0)}$ since both $\boldsymbol{\gamma}^{[0]}$ and $\boldsymbol{\gamma}_{[0]}$ go to $\mathbf{0}$ when $t' = t$.

d. Higher order [ ] quantities are obtained by successive *partial* differentiation with respect to t' with $\boldsymbol{r}$, t being held constant. Higher order ( ) quantities are obtained by successive *convected* differentiation.

e. The ( ) quantities are easily obtained from the velocity field since the convected differentiation operators involve only $(\boldsymbol{v}\cdot\nabla)$, $\nabla\boldsymbol{v}$, and $(\nabla\boldsymbol{v})^{\dagger}$. The [ ] quantities require a knowledge of the displacement functions, which are needed to get the displacement gradient tensors $\boldsymbol{\Delta}$ and $\mathbf{E}$.

f. In the limit of very small deformations (i.e., linear viscoelasticity), the relative strain tensors, $\boldsymbol{\gamma}^{[0]}$ and $\boldsymbol{\gamma}_{[0]}$, both simplify to the infinitesimal strain tensor $\boldsymbol{\gamma}$ as pointed out earlier. In addition in this limit the tensors $\boldsymbol{\gamma}^{[n]}$ and $\boldsymbol{\gamma}_{[n]}$ both simplify to $\partial^n\boldsymbol{\gamma}/\partial t'^n$, and similarly the tensors $\boldsymbol{\gamma}^{(n)}$ and $\boldsymbol{\gamma}_{(n)}$ both simplify to $\partial^n\boldsymbol{\gamma}/\partial t^n$.

g. In Appendix C the most important kinematic tensors are tabulated for homogeneous shear flows and shearfree flows, that is, for those flows encountered in material-function measurements. Table C.1 is particularly useful for evaluating constitutive equations. Appendix B contains tables of the components of various kinematic tensors in cylindrical and spherical coordinates.

Derivatives of Stress in Convected Coordinate Systems

It is possible to show that the same covariant and contravariant derivatives which have been defined in Eqs. (10) and (11), can be used for **stress tensor**.

Covariant Convected Derivative of Stress (Lower Convected Derivative):

$$\begin{aligned}
\boldsymbol{\tau}^{(1)} &= \frac{D\boldsymbol{\tau}}{Dt} + \left\{ (\nabla \boldsymbol{V}) \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot (\nabla \boldsymbol{V})^{T} \right\} \\
&\vdots \\
\boldsymbol{\tau}^{(n)} &= \frac{D\boldsymbol{\tau}^{(n-1)}}{Dt} + \left\{ (\nabla \boldsymbol{V}) \cdot \boldsymbol{\tau}^{(n-1)} + \boldsymbol{\tau}^{(n-1)} \cdot (\nabla \boldsymbol{V})^{T} \right\}
\end{aligned} \tag{12}$$

Contravariant Convected Derivative of Stress (Upper Convected Derivative)

$$\begin{aligned}
\boldsymbol{\tau}_{(1)} &= \frac{D\boldsymbol{\tau}}{Dt} - \left\{ (\nabla \boldsymbol{V})^{T} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot (\nabla \boldsymbol{V}) \right\} \\
&\vdots \\
\boldsymbol{\tau}_{(n)} &= \frac{D\boldsymbol{\tau}_{(n-1)}}{Dt} - \left\{ (\nabla \boldsymbol{V})^{T} \cdot \boldsymbol{\tau}_{(n-1)} + \boldsymbol{\tau}_{(n-1)} \cdot (\nabla \boldsymbol{V}) \right\}
\end{aligned} \tag{13}$$

Methods of Developing the Constitutive Equations

Constitutive equations have been generated in a number of ways:

1. One can replace the tensors occurring in linear viscoelasticity by the appropriate [ ] and ( ) quantities defined in this chapter to generate "quasi-linear constitutive equations." There is, however, no unique way to do this. For example, the linear Jeffreys model of Eq. 5.2-9 could be transformed into a nonlinear viscoelastic equation in a number of ways; two possibilities are:

Jeffreys linear Model:

$$\tau+\lambda_1\tau=\eta\left(\gamma+\lambda_2\gamma\right)$$

Oldroyd-A Model

$$\tau+\lambda_1\tau^{(1)}=\eta\left(\gamma^{(1)}+\lambda_2\gamma^{(2)}\right) \rightarrow \Psi_1>0 \;\&\; \Psi_2=-\Psi_1 \qquad (14a)$$

Oldroyd-B Model

$$\tau+\lambda_1\tau_{(1)}=\eta\left(\gamma_{(1)}+\lambda_2\gamma_{(2)}\right) \rightarrow \Psi_1>0 \;\&\; \Psi_2=0 \qquad (14b)$$

Only comparison with experimental data or molecular theories can enable one to choose between these two models. Neither model gives a non-Newtonian viscosity or shear-rate dependent normal stress coefficients in steady-state shear flow. Equation (14a) gives $\Psi_2=-\Psi_1$, whereas (14b) gives $\Psi_2=0$; the latter is somewhat closer to the experimental facts, and therefore (14b) is preferred.

2. One can put together empirical expressions, using the [ ] and ( ) quantities, in a wide variety of combinations, linear or nonlinear, differential or integral. The models of Oldroyd and Giesekus (Chapter 7) and those of K–BKZ and Rivlin and Sawyers (Chapter 8) are examples of this. Here one proceeds on a try-it-and-see basis until combinations are obtained that are capable of describing experimental data on material functions to the degree desired.

3. One can proceed in a mathematical fashion and make various kinds of ordered expansions. One such attempt leads to the "retarded-motion expansion" discussed in Chapter 6.

4. One can attempt to see how the ordered expansions in (3) simplify for various special categories of flows. An example of this is the CEF equation, for steady shear flows, discussed later in this chapter.

5. One can make use of molecular theories; these theories, of course, involve making some kind of model for the macromolecules in the fluid and some kind of assumptions as to how these molecules interact with one another. If the molecular modeling is sufficiently simple, a complete constitutive equation can be obtained. However, because of the crudeness of the modeling, the final constitutive equation may not be very realistic. If more faithful modeling is done at the molecular level, then it may not even be possible to work all the way through to a constitutive equation, or if it is possible, the resulting constitutive equation may be too complicated for general use in hydrodynamic calculations. In any case, the molecular theories have provided some very useful ideas as to what kinds of terms ought to be included in constitutive equations. Molecular theories are discussed extensively in Volume 2.

Some Points about the book of "Dynamics of Polymeric Liquids" by Bird *et al.*

1. In This book, the symbols $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ are used as the total and shear momentum flux! They can be changed to the total stress and shear stress by considering $-\boldsymbol{\sigma}$ and $-\boldsymbol{\tau}$ instead of $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ in equations of this book, respectively. For Example:

$$\text{In the book of Bird } et\ al.\text{:} \begin{cases} \boldsymbol{\tau} = -\eta\dot{\boldsymbol{\gamma}} \\ \boldsymbol{\pi} = p\mathbf{I} + \boldsymbol{\tau} \end{cases} \longrightarrow \text{You can write:} \begin{cases} \boldsymbol{\tau} = \eta\dot{\boldsymbol{\gamma}} \\ \boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau} \end{cases} \tag{15}$$

2. Some symbols and operators of this book are:

$$\begin{aligned} &\mathbf{A} : \mathbf{B} = tr\{\mathbf{A}.\mathbf{B}\} \\ &\mathbf{A}^{\dagger} = \text{Transposed of } \mathbf{A} = \mathbf{A}^{T} \end{aligned} \tag{16}$$

3. The covariant and contravariant convected derivatives of any tensor are known mostly in literature as the lower and upper convected derivatives, respectively. The common symbols for these derivations are Δ and ∇. For example:

$$\begin{aligned} &\boldsymbol{\tau}^{(1)} = \overset{\Delta}{\boldsymbol{\tau}} \quad \& \quad \dot{\boldsymbol{\gamma}}^{(2)} = \overset{\Delta}{\dot{\boldsymbol{\gamma}}} \\ &\boldsymbol{\tau}_{(1)} = \overset{\nabla}{\boldsymbol{\tau}} \quad \& \quad \dot{\boldsymbol{\gamma}}_{(2)} = \overset{\nabla}{\dot{\boldsymbol{\gamma}}} \end{aligned} \tag{17}$$


The End
And The Beginning