STEADY TWO- AND THREE-DIMENSIONAL PROBLEMS. SEPARATION OF VARIABLES. ORTHOGONAL FUNCTIONS.

Thus far our discussion has been confined to steady one-dimensional problems. Since these yield ordinary differential equations of second order, they are, in principle, always solvable. On the other hand, steady two- and three-dimensional and unsteady problems result in partial differential equations for which no general method of solution is available. In this chapter and in Chapters 5, 6, and 7, some methods for exact solution of these problems will be developed.

When the boundaries of a multidimensional conduction problem correspond to the coordinate surfaces in a system of orthogonal coordinates, such as cartesian, cylindrical, or spherical coordinates, an exact solution by analytical methods becomes possible. One common method is based on the *separation of variables,* another on the *Laplace transforms* of the operational calculus. There are convincing arguments as to which of these two most-used methods might be better suited for a particular problem; however, the Laplace transforms, though convenient for the solution of complicated problems, require the knowledge of more advanced mathematics. Thus, delaying the Laplace transforms to Chapter 7, we devote this chapter to the method of separation of variables. In the next four sections we review the mathematics necessary for this method.

4–1. Boundary-Value Problems. Characteristic-Value Problems

Consider first an ordinary differential equation of second order which may result from the differential formulation of a steady one-dimensional conduction problem. The solution of this equation involves two arbitrary constants which are determined by two conditions, each specified at one boundary of the problem. Problems of this type are called *boundary-value problems* to distinguish them from *initial-value problems,* in which all conditions are specified at one location. An example of an initial-value problem is the free fall of a body, mentioned in Chapter 1.†

Next consider the homogeneous‡ linear equation of second order

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + f_2(x)y = 0, \tag{4–1}$$

which vanishes at the boundaries $x = a$ and $x = b$,

$$y(a) = 0, \qquad y(b) = 0. \tag{4–2}$$

† An unsteady problem becomes an initial-value problem when it is lumped, and an initial- and boundary-value problem when it is distributed.

‡ See Section 3–4 for the definition of homogeneous and nonhomogeneous differential equations and boundary conditions.

The general solution of this equation may be written in the form

$$y = C_1 y_1(x) + C_2 y_2(x), \tag{4-3}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions, and C_1 and C_2 are arbitrary constants. This solution, combined with boundary conditions, gives

$$\begin{aligned} C_1 y_1(a) + C_2 y_2(a) &= 0, \\ C_1 y_1(b) + C_2 y_2(b) &= 0. \end{aligned} \tag{4-4}$$

One possible solution of these homogeneous equations is $C_1 = C_2 = 0$, leading to the *trivial solution* $y \equiv 0$. If the determinant of the coefficients of C_1 and C_2 does not vanish, then this is the *only solution*. If the determinant of the coefficients does vanish, then

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1(b) & y_2(b) \end{vmatrix} = 0. \tag{4-5}$$

Now the two equations of (4-4) become identical and one constant can be expressed as a multiple of the other by use of either equation, the second constant being arbitrary. Thus *if Eq.* (4-5) *is satisfied* and if we discard, for example, the second equation of (4-4), we obtain from the first equation $C_2 y_2(a) = -C_1 y_1(a)$. Defining a new constant C by $C_1 = C y_2(a)$ yields $C_2 = -C y_1(a)$, and Eq. (4-3) becomes

$$y = C[y_2(a) y_1(x) - y_1(a) y_2(x)]. \tag{4-6}$$

It may readily be seen that Eq. (4-6) satisfies the boundary conditions. One condition, $y(a) = 0$, is obtained directly from Eq. (4-6). For the other, $y(b) = 0$, Eq. (4-6) gives $y_2(a)y_1(b) - y_1(a)y_2(b) = 0$, which is the negative of Eq. (4-5). It should be remarked that Eq. (4-6) is a nontrivial solution only if $y_1(a)$ and $y_2(a)$ are not both zero. If $y_1(a) = y_2(a) = 0$, the first equation of (4-4) is a trivial identity, in which case only the second equation may be used to relate C_1 and C_2. This gives a nontrivial solution in the form

$$y = C[y_2(b) y_1(x) - y_1(b) y_2(x)],$$

provided $y_1(b)$ and $y_2(b)$ are not both zero. If $y_1(x)$ and $y_2(x)$ both vanish at $x = a$ and $x = b$, then Eq. (4-3) satisfies Eq. (4-2) for arbitrary values of both C_1 and C_2. If Eq. (4-5) is not satisfied, the only solution is the trivial one $y \equiv 0$.

The solution of two- and three-dimensional steady conduction problems, and that of one- and multidimensional unsteady problems may be reduced to the solution of Eq. (4-1), whose coefficients $f_1(x)$ and/or $f_2(x)$ depend on a parameter λ. In such problems the determinant of Eq. (4-5) may vanish only for certain values of λ, say $\lambda_1, \lambda_2, \lambda_3, \ldots$; these values are called the *characteristic values*. For each such value of λ a solution similar to Eq. (4-6) is obtained; these particular solutions are the *characteristic functions* of the problem, and problems of this kind are known as *characteristic-value problems.* The terms *eigenvalues, eigenfunctions,* and *eigenvalue problems* are also used frequently.

The foregoing generalized procedure can best be explained by an illustrative example. Reconsider the differential equation

$$\frac{d^2y}{dx^2} + y = 0$$

which was used in Section 3–6 to show the method of power series solution of differential equations. Furthermore, assume that this homogeneous equation involves a parameter λ as

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0, \tag{4–7}$$

and is subject to homogeneous boundary conditions

$$y(0) = 0, \tag{4–8}$$

$$y(L) = 0. \tag{4–9}$$

The general solution of Eq. (4–7) is

$$y = C_1 \sin \lambda x + C_2 \cos \lambda x. \tag{4–10}$$

The use of Eq. (4–8) results in $C_2 = 0$ and

$$y = C_1 \sin \lambda x, \tag{4–11}$$

and Eq. (4–9), combined with Eq. (4–11), gives $0 = C_1 \sin \lambda L$. The problem has nontrivial solutions only if λ satisfies the equation $0 = \sin \lambda L$. Therefore,

$$\lambda_n = n\pi/L, \qquad n = 1, 2, 3, \ldots, \tag{4–12}$$

and the corresponding solutions of Eq. (4–11) are

$$y = C_1\varphi_n(x), \qquad \varphi_n(x) = \sin\,(n\pi/L)x. \tag{4–13}$$

Note that no new solutions are obtained when n assumes negative integer values.

Thus the foregoing boundary-value problem, Eqs. (4–7), (4–8), and (4–9), has no solution other than the trivial solution $y \equiv 0$, unless λ assumes one of the characteristic values given by Eq. (4–12). Corresponding to each characteristic value of λ_n there exists a characteristic function $\varphi_n(x)$ given by Eq. (4–13), such that any constant multiple of this function is a solution of the problem. It is important to note that the boundary-value problem given by

$$\frac{d^2y}{dx^2} - \lambda^2 y = 0; \qquad y(0) = 0, \quad y(L) = 0$$

has no solution other than the trivial solution $y \equiv 0$ corresponding to $\lambda = 0$. Hence there does not exist any set of characteristic values and characteristic functions for this problem. This illustrates the fact that *a boundary-value problem may or may not be a characteristic-value problem. A boundary-value problem is a characteristic-value problem when it has particular solutions that are periodic in nature; the period and amplitude of these solutions may or may not be constant.* Examples are the circular functions and Bessel functions of the first and second kinds, of any order. Since the starting point of a characteristic-value problem is a boundary-value problem, *a characteristic-value problem is always a boundary-value problem.*

In the next three sections the general properties of characteristic functions are investigated.

4-2. Orthogonality of Characteristic Functions

By definition, two functions $\varphi_n(x)$ and $\varphi_m(x)$ are said to be *orthogonal with respect to a weighting function* $w(x)$, over a finite interval (a, b), if the integral of the product $w\varphi_n\varphi_m$ over that interval vanishes as

$$\int_a^b w(x)\varphi_n(x)\varphi_m(x)\,dx = 0, \qquad m \neq n. \tag{4-14}$$

Furthermore, a set of functions is said to be *orthogonal* in (a, b) if all pairs of distinct functions in the set are orthogonal in (a, b). The word *orthogonality* comes from vector analysis. Let $\boldsymbol{\varphi}_m(x_i)$ denote a vector in three-dimensional space whose rectangular components are $\varphi_m(x_1)$, $\varphi_m(x_2)$, and $\varphi_m(x_3)$. Two vectors, $\boldsymbol{\varphi}_m(x_i)$ and $\boldsymbol{\varphi}_n(x_i)$, are said to be orthogonal, or perpendicular to each other, if

$$\boldsymbol{\varphi}_m(x_i)\cdot\boldsymbol{\varphi}_n(x_i) = \sum_{i=1}^{3}\varphi_m(x_i)\varphi_n(x_i) = 0.$$

When the units of length on the coordinate axes vary from one axis to another, the foregoing scalar product assumes the form

$$\boldsymbol{\varphi}_m(x_i)\cdot\boldsymbol{\varphi}_n(x_i) = \sum_{i=1}^{3} w(x_i)\varphi_m(x_i)\varphi_n(x_i),$$

where the *weighting numbers* $w(x_1)$, $w(x_2)$, and $w(x_3)$ depend upon the units of length used along the three axes. Similarly, the vectors in an N-dimensional space having components $\varphi_m(x_i)$, $\varphi_n(x_i)$, $i = 1, 2, 3, \ldots, N$ are said to be orthogonal with respect to the weighting numbers $w(x_i)$ when

$$\boldsymbol{\varphi}_m(x_i)\cdot\boldsymbol{\varphi}_n(x_i) = \sum_{i=1}^{N} w(x_i)\varphi_m(x_i)\varphi_n(x_i) = 0. \tag{4-15}$$

If the vector space has an infinite number of dimensions, the components $\varphi_m(x_i)$ and $\varphi_n(x_i)$ become continuously distributed and x_i is no longer a discrete number but a continuous variable, say x; in this case Eq. (4-15) becomes identical to Eq. (4-14).

It will now be shown that *the characteristic functions of a characteristic-value problem are orthogonal over a finite interval with respect to a weighting function.* To establish this fact, consider the characteristic-value problem composed of the linear homogeneous second-order differential equation of the general form

$$\frac{d^2y}{dx^2} + f_1(x)\frac{dy}{dx} + [f_2(x) + \lambda^2 f_3(x)]y = 0$$

and two homogeneous boundary conditions prescribed at the ends of the finite interval (a, b). This equation, multiplied through by the factor $\exp\left[\int f_1(x)\,dx\right] = p(x)$ and with the functions defined as $f_2(x)p(x) = q(x)$ and $f_3(x)p(x) = w(x)$, may be rearranged in the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda^2 w(x)]y = 0, \tag{4-16}$$

which is more convenient for the following discussion.

Let λ_m, λ_n be any two distinct characteristic numbers, that is, $m \neq n$, and let $\varphi_m(x)$, $\varphi_n(x)$ be the corresponding characteristic functions. Since $y = \varphi_m(x)$ and $y = \varphi_n(x)$ are solutions of Eq. (4-16),

$$\frac{d}{dx}\left(p\frac{d\varphi_m}{dx}\right) + (q + \lambda_m^2 w)\varphi_m = 0,$$

$$\frac{d}{dx}\left(p\frac{d\varphi_n}{dx}\right) + (q + \lambda_n^2 w)\varphi_n = 0.$$

Multiplying the first equation by φ_n and the second by φ_m, then subtracting the second of the resulting equations from the first one gives

$$\varphi_n\frac{d}{dx}\left(p\frac{d\varphi_m}{dx}\right) - \varphi_m\frac{d}{dx}\left(p\frac{d\varphi_n}{dx}\right) + (\lambda_m^2 - \lambda_n^2)w\varphi_m\varphi_n = 0.$$

Integrating this equation over the finite interval (a, b) yields

$$(\lambda_n^2 - \lambda_m^2)\int_a^b w\varphi_m\varphi_n\,dx = \int_a^b\left[\varphi_n\frac{d}{dx}\left(p\frac{d\varphi_m}{dx}\right) - \varphi_m\frac{d}{dx}\left(p\frac{d\varphi_n}{dx}\right)\right]dx,$$

and integration by parts for the right-hand member results in

$$(\lambda_n^2 - \lambda_m^2)\int_a^b w\varphi_m\varphi_n\,dx = \left\{p(x)\left[\varphi_n(x)\frac{d\varphi_m(x)}{dx} - \varphi_m(x)\frac{d\varphi_n(x)}{dx}\right]\right\}\Bigg|_a^b. \tag{4-17}$$

Since both $y = \varphi_m(x)$ and $y = \varphi_n(x)$ are particular solutions of Eq. (4–16), the right-hand side of Eq. (4–17) vanishes when one of the following conditions is prescribed at each end of the interval (a, b):

$$y = 0, \tag{4–18}$$

$$dy/dx = 0, \tag{4–19}$$

$$dy/dx + By = 0, \tag{4–20}$$

where B is an arbitrary parameter. The fact that Eq. (4–17) vanishes when Eq. (4–20) is satisfied may be clarified by rearranging the right-hand member of Eq. (4–17) in the form

$$\varphi_n\varphi'_m - \varphi_m\varphi'_n = \varphi_n\varphi'_m - \varphi_m\varphi'_n \pm B\varphi_m\varphi_n = \varphi_n(\varphi'_m + B\varphi_m) - \varphi_m(\varphi'_n + B\varphi_n).$$

Particularly, if $p(x) = 0$ when $x = a$ or $x = b$, the right-hand side of Eq. (4–17) vanishes, and the condition given by Eq. (4–18), (4–19), or (4–20) satisfied at $x = a$ or $x = b$ can be dropped from the problem provided y and (dy/dx) are finite at that point. If $p(b) = p(a)$, the orthogonality continues to exist when the boundary conditions are replaced by the conditions $y(b) = y(a)$ and $y'(b) = y'(a)$, which are called the *periodic boundary conditions.*

As an example, reconsider the characteristic-value problem given by Eqs. (4–7), (4–8), and (4–9). Comparison of Eqs. (4–7) and (4–16) gives $w(x) = 1$, and the condition of orthogonality for this problem is

$$\int_0^L \varphi_m(x)\varphi_n(x)\,dx = \int_0^L \sin(m\pi x/L)\sin(n\pi x/L)\,dx = 0, \qquad m \neq n,$$

which can also be verified independently by direct integration.

4–3. Expansion of Arbitrary Functions in Series of Orthogonal Functions

Let $\varphi_n(x)$ be a set of functions orthogonal with respect to a weighting function $w(x)$ over a finite interval (a, b). We wish to expand an arbitrary function $f(x)$ into a series of this set as

$$f(x) = b_0\varphi_0(x) + b_1\varphi_1(x) + b_2\varphi_2(x) + \cdots = \sum_{n=0}^{\infty} b_n\varphi_n(x). \tag{4–21}$$

Assuming that such an expansion exists, we may evaluate the coefficients b_n. Multiplying both sides of Eq. (4–21) by $w(x)\varphi_m(x)$ and integrating the result over the interval (a, b) with the *assumption* that the integral of the infinite sum is equivalent to the sum of the integrals, we have

$$\int_a^b w(x)f(x)\varphi_m(x)\,dx = \sum_{n=0}^{\infty} b_n \int_a^b w(x)\varphi_m(x)\varphi_n(x)\,dx, \tag{4–22}$$

where $\varphi_m(x)$ is the mth term in the set. Using the orthogonality of the set, we find that all terms in the sum on the right of Eq. (4–22) are zero except the term corresponding to $n = m$. Hence Eq. (4–22) gives

$$b_n = \frac{\int_a^b w(x)f(x)\varphi_n(x)\,dx}{\int_a^b w(x)\varphi_n^2(x)\,dx}. \tag{4–23}$$

The general problem of determining whether the expansion given by Eq. (4–21) represents the function $f(x)$ is beyond the scope of this text. Let us note, however, the generality of such expansions. Unlike the Taylor and Maclaurin series expansions of a function $f(x)$, which require that the function and all its derivatives be continuous, if $f(x)$ is piecewise differentiable† in a finite interval (a, b), the series given by Eq. (4–21) converges to $f(x)$ at all points of continuity, and to the mean value $\frac{1}{2}[f(x+) + f(x-)]$ of $f(x)$ at any discontinuity.

In the next section we show that the well-known Fourier series actually are special cases of the expansion in terms of a set of orthogonal functions involving circular functions.

4–4. Fourier Series

We have learned in Section 4–1 that the boundary-value problem given by

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0; \qquad y(0) = 0, \quad y(L) = 0$$

leads to a characteristic-value problem which has the following characteristic values and functions

$$\lambda_n = n\pi/L, \qquad n = 1, 2, 3, \ldots, \tag{4–12}$$

$$\varphi_n = \sin{(n\pi/L)x}. \tag{4–13}$$

We now wish to expand an arbitrary function $f(x)$ into a series of these characteristic functions as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin{(n\pi/L)x}, \qquad 0 < x < L, \tag{4–24}$$

where the coefficients b_n are calculated from Eq. (4–23):

$$b_n = \frac{\int_0^L f(x)\sin{(n\pi/L)x}\,dx}{\int_0^L \sin^2{(n\pi/L)x}\,dx}. \tag{4–25}$$

It can readily be shown by direct integration that the denominator of Eq. (4–25)

† A function $f(x)$ is said to be piecewise differentiable over a finite range if it is possible to divide that range into a finite number of intervals such that $f(x)$ is differentiable inside each interval and $f'(x)$ approaches finite values from either side of a discontinuity.

is equal to $L/2$ regardless of the value of n. Hence

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx. \tag{4-26}$$

The series given by Eq. (4-24) is the definition of the *Fourier sine series* of $f(x)$ over the interval $(0, L)$. All terms of this series are periodic with the period $2L$, which is twice the length of the interval. If x is replaced by $-x$, the sign of each term is reversed. Hence in the interval $(-L, 0)$ the series represents the function $-f(-x)$. The behavior of the series in the interval $(-L, L)$ is repeated periodically for all values of x. If $f(x)$ is an odd function of x, the series given by Eq. (4-24) represents $f(x)$ not only in the interval $(0, L)$ but also in the interval $(-L, L)$. If, in addition, $f(x)$ is periodic, of period $2L$, the series represents $f(x)$ everywhere.

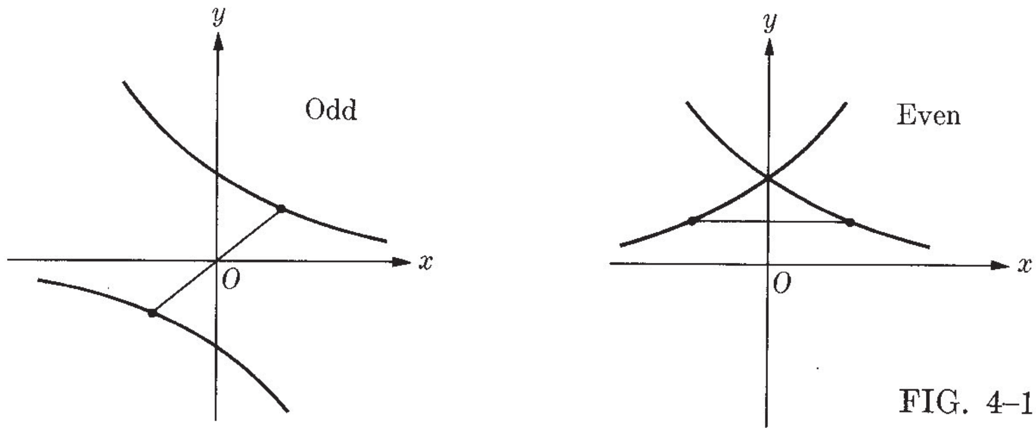

FIG. 4-1

A function $f(x)$ is said to be an *odd function* if $f(-x) = -f(x)$ and an *even function* if $f(-x) = f(x)$. An odd function is symmetrical with respect to the origin, whereas an even function is symmetrical with respect to the y-axis (Fig. 4-1).

Consider, as an example, the Fourier sine series of the function

$$f(x) = \begin{cases} 0, & -\infty < x < 0 \quad \text{and} \quad L/2 < x < \infty \\ 1, & 0 < x < L/2 \end{cases} \tag{4-27}$$

over the interval $(0, L)$ (Fig. 4-2). The coefficients of the series are

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx = \frac{2}{L}\int_0^{L/2} 1\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right);$$

hence the series is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)x. \tag{4-28}$$

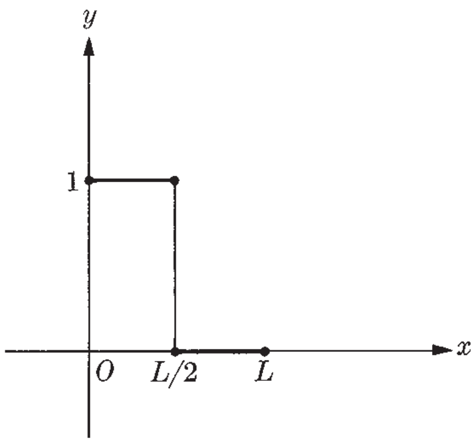

FIG. 4–2

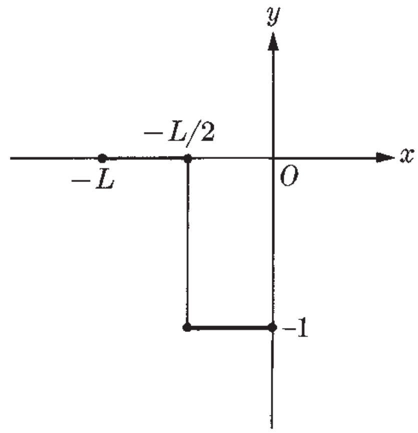

FIG. 4–3

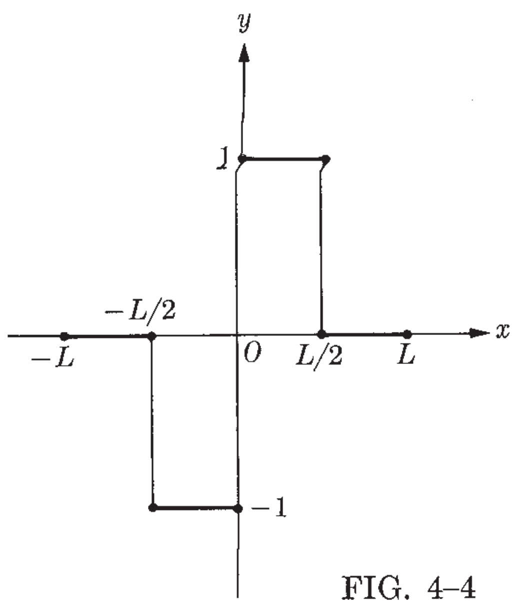

FIG. 4–4

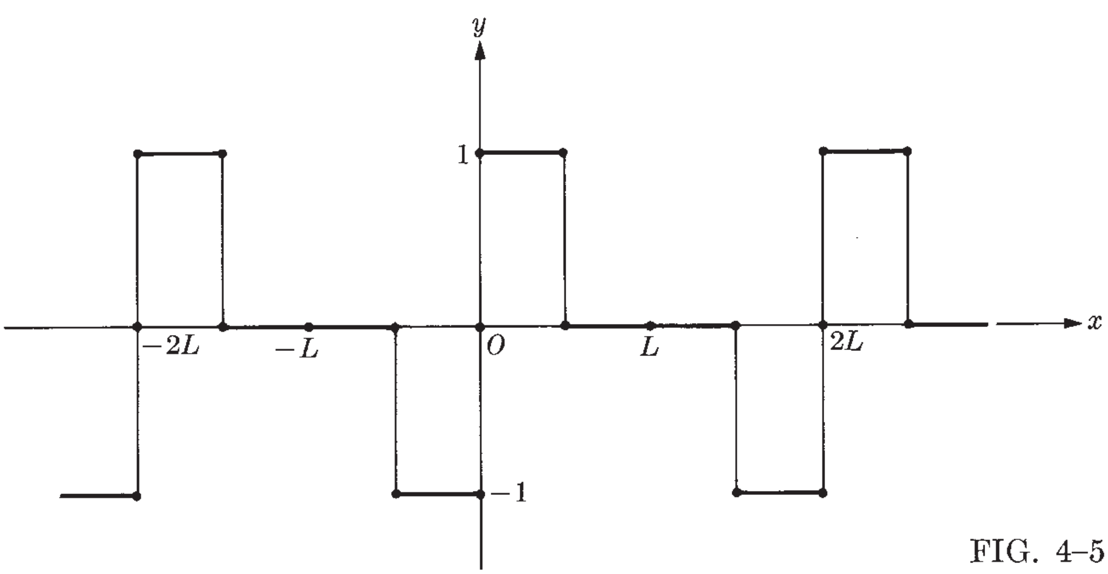

FIG. 4–5

The same series also represents the function

$$-f(x) = f(-x) = \begin{cases} 0, & -\infty < x < -L/2 \quad \text{and} \quad 0 < x < \infty \\ -1, & -L/2 < x < 0 \end{cases}$$

over the interval $(-L, 0)$ (Fig. 4–3). Thus the Fourier sine series of the odd function defined by

$$f(x) = \begin{cases} -1, & -L/2 < x < 0 \\ 0, & -\infty < x < L/2 \quad \text{and} \quad L/2 < x < \infty \\ 1, & 0 < x < L/2 \end{cases}$$

represents $f(x)$ in the interval $(-L, L)$ (Fig. 4–4) except at points of discontinuity, and repeats this behavior periodically for all values of x (Fig. 4–5).

A series expansion involving cosine terms rather than sine terms may be obtained by considering instead the boundary-value problem

$$\frac{d^2y}{dx^2} + \lambda^2 y = 0; \qquad y'(0) = 0, \quad y'(L) = 0. \tag{4–29}$$

Following the procedure of the previous characteristic-value problem, we may readily obtain the characteristic values and functions in the form

$$\lambda_n = n\pi/L, \quad n = 0, 1, 2, \ldots; \qquad \varphi_n = \cos (n\pi/L)x. \tag{4–30}$$

Note that $\varphi_0(x) = 1$ is a member of the set of characteristic functions given in Eq. (4–30) corresponding to $\lambda_0 = 0$. Now, expanding an arbitrary function $f(x)$ into a series of the foregoing set, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos (n\pi/L)x, \qquad 0 < x < L, \tag{4–31}$$

where the coefficients a_n are calculated from Eq. (4–23) in the form

$$a_0 = \frac{\int_0^L f(x)\, dx}{\int_0^L dx}, \qquad a_n = \frac{\int_0^L f(x) \cos (n\pi/L)x\, dx}{\int_0^L \cos^2 (n\pi/L)x\, dx}. \tag{4–32}$$

Furthermore, noting that

$$\int_0^L \cos^2 \left(\frac{n\pi}{L}\right) x\, dx = \begin{cases} L, & n = 0 \\ L/2, & n = 1, 2, 3, \ldots \end{cases}$$

we may rearrange Eq. (4–32) as

$$a_0 = \frac{1}{L} \int_0^L f(x)\, dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \left(\frac{n\pi}{L}\right) x\, dx. \tag{4–33}$$

The series given by Eq. (4–31) is the definition of the *Fourier cosine series* of $f(x)$ over the interval $(0, L)$. All terms of Eq. (4–31) are even functions of x, and they are periodic with the period $2L$. This series represents the function $f(-x)$ in the interval $(-L, 0)$. If $f(x)$ is an even function of x, the series converges to $f(x)$ not only in the interval $(0, L)$ but also in the interval $(-L, 0)$. If, in addition, $f(x)$ is periodic, of period $2L$, the series represents $f(x)$ everywhere.

Consider now the Fourier cosine series of the previous example, Eq. (4–27). The coefficients of the series are

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_0^{L/2} 1 \cdot dx = \tfrac{1}{2},$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{2}{L}\int_0^{L/2} 1\cdot\cos\left(\frac{n\pi}{L}\right)x\,dx$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

Therefore,

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{L}\right)x. \tag{4–34}$$

The same series also represents the function

$$f(x) = f(-x) = \begin{cases} 0, & -\infty < x < -L/2 \quad \text{and} \quad 0 < x < \infty \\ -1, & -L/2 < x < 0 \end{cases}$$

over the interval $(-L, 0)$ (Fig. 4–6). Thus the Fourier cosine series of the even function defined by

$$f(x) = \begin{cases} 0, & -\infty < x < -L/2 \quad \text{and} \quad L/2 < x < \infty \\ 1, & -L/2 < x < L/2 \end{cases}$$

represents $f(x)$ in the interval $(-L, L)$ (Fig. 4–7) except at points of discontinuity, and repeats this behavior periodically for all values of x (Fig. 4–8).

So far, we have seen that any piecewise continuous function can be expressed in the interval $(0, L)$ by a series consisting of sines or cosines with the common period $2L$. When the function is odd, the sine series representation is valid in the interval $(-L, L)$, whereas for an even function the cosine series representation holds in the same interval.

We now wish to express a function $f(x)$, which is piecewise continuous in the interval $(-L, L)$, in terms of both sines and cosines having the common

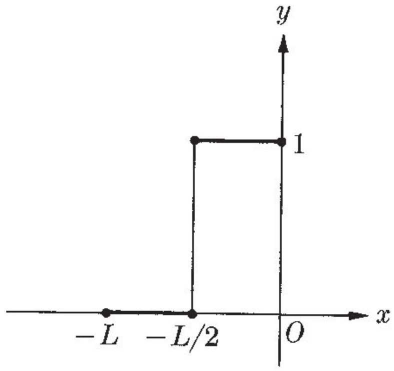

FIG. 4-6

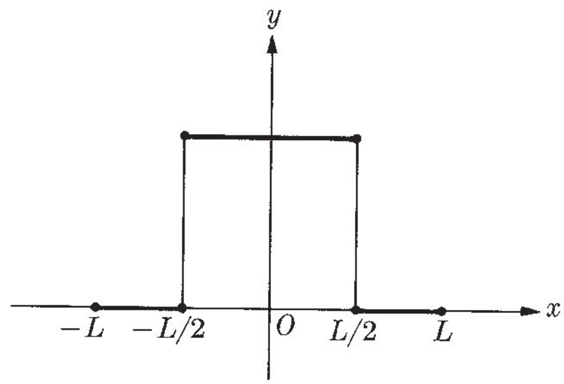

FIG. 4-7

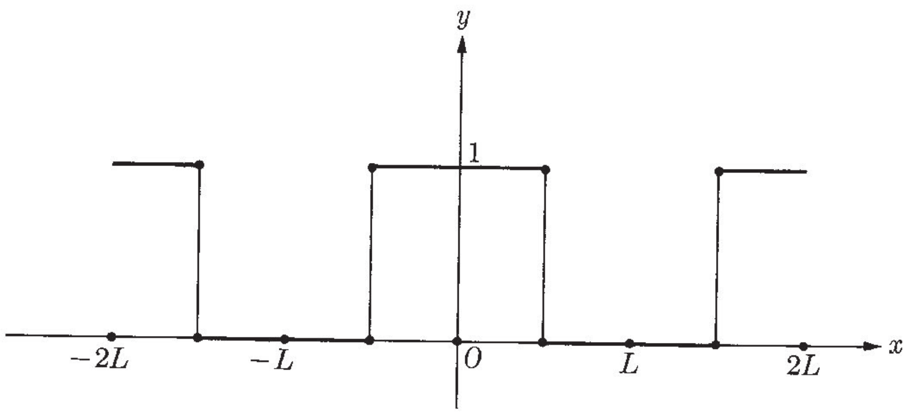

FIG. 4-8

period $2L$. Noting that

$$f(x) = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)],$$

where the function in the first brackets is even and that in the second is odd, we arrive at the fact that an arbitrary function can be expressed as the sum of an even function and an odd function. Hence

$$f(x) = f_e(x) + f_o(x). \tag{4-35}$$

Expressing $f_e(x)$ in terms of cosines and $f_o(x)$ in terms of sines in the interval $(-L, L)$, we have

$$f_e(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos (n\pi/L)x, \tag{4-36}$$

$$f_o(x) = \sum_{n=1}^{\infty} b_n \sin (n\pi/L)x, \tag{4-37}$$

where

$$a_0 = \frac{1}{L}\int_0^L f_e(x)\,dx, \qquad a_n = \frac{2}{L}\int_0^L f_e(x)\cos\left(\frac{n\pi}{L}\right)x\,dx,$$

$$b_n = \frac{2}{L}\int_0^L f_o(x)\sin\left(\frac{n\pi}{L}\right)x\,dx.$$

Since the integrands of these equations are even functions of x, replacing $\int_0^L$ by $\frac{1}{2}\int_{-L}^L$ gives

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx. \tag{4-38}$$

It is clear that the odd part of the first two integrals and the even part of the third integral vanish identically. Hence introducing Eqs. (4–36) and (4–37) into Eq. (4–35) yields

$$f(x) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\pi/L)x + b_n \sin(n\pi/L)x], \qquad -L < x < L. \tag{4-39}$$

This series is the definition of the *complete Fourier series* of $f(x)$ over the interval $(-L, L)$. When $f(x)$ is even, the resulting series involves cosine terms only; a series involving only sines results when $f(x)$ is odd. Otherwise Eq. (4–39) is in terms of both sines and cosines of period $2L$ in the interval $(-L, L)$, and repeats this behavior periodically for all values of x.

Finally, let us find the complete Fourier series of the function given by Eq. (4–27). The coefficients of this series are

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_0^{L/2} 1\cdot dx = \tfrac{1}{4},$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{L}\int_0^{L/2} 1\cdot\cos\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{n\pi}\sin\frac{n\pi}{2},$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$
$$= \frac{1}{L}\int_0^{L/2} 1\cdot\sin\left(\frac{n\pi}{L}\right)x\,dx = \frac{1}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

Therefore,

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{L}\right)x + \left(1 - \cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)x\right], \tag{4-40}$$

which converges for all values of x, in the sense described, to the periodic function shown in Fig. 4–9. [Compare Eq. (4–40) with the sum of the sine and cosine series expansions of the same function obtained by adding Figs. (4–5) and (4–8) geometrically, or Eqs. (4–28) and (4–34) algebraically.]

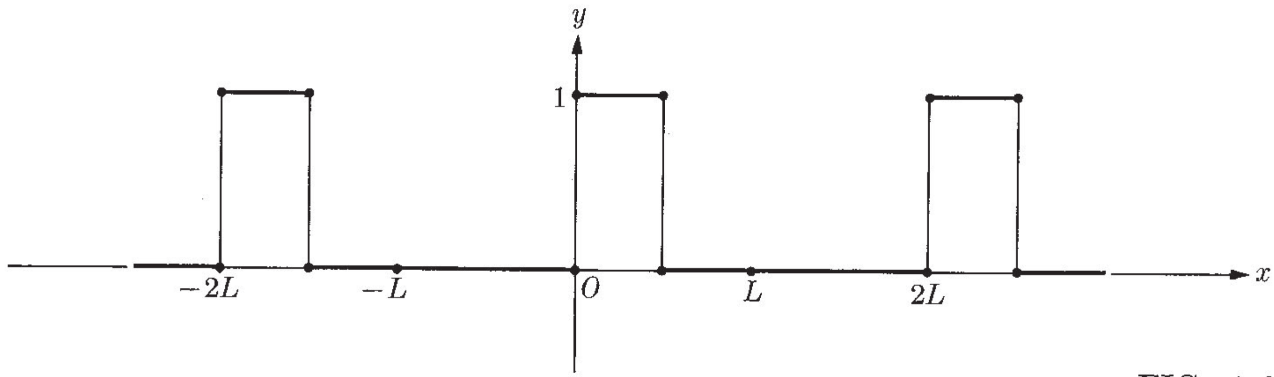

FIG. 4–9

Having gained the foregoing mathematical background we now proceed to the solution of problems by the method of separation of variables.

4–5. Separation of Variables. Steady Two-Dimensional Cartesian Geometry

When the boundary conditions of a problem are in terms of specified T, $\partial T/\partial n$, or $\partial T/\partial n + BT$, where n is the normal to the boundary and B a constant, the solution may be expressed as a product of functions of each coordinate separately. This allows the boundary conditions to be expressed in terms of a single variable, and reduces the partial differential equation to a set of ordinary differential equations.

The essential features of the method will now be illustrated by means of a steady two-dimensional example. Consider the second-order partial differential equation

$$a_1(x)\frac{\partial^2 T}{\partial x^2} + a_2(x)\frac{\partial T}{\partial x} + a_3(x)T + b_1(y)\frac{\partial^2 T}{\partial y^2} + b_2(y)\frac{\partial T}{\partial y} + b_3(y)T = 0. \tag{4–41}$$

A more generalized form of this equation which involves coefficients as functions of both independent variables is not suitable for the separation of variables.

Assume the existence of a product solution

$$T(x, y) = X(x)Y(y), \tag{4–42}$$

where X is a function of x alone and Y is a function of y. This assumption becomes meaningful when the two functions X and Y actually satisfy separate differential equations.

Introducing Eq. (4–42) into Eq. (4–41) and dividing the result by XY yields

$$\left[a_1(x)\frac{d^2X}{dx^2} + a_2(x)\frac{dX}{dx} + a_3X\right]\frac{1}{X} = -\left[b_1(y)\frac{d^2Y}{dy^2} + b_2(y)\frac{dY}{dy} + b_3(y)Y\right]\frac{1}{Y}. \tag{4–43}$$