WIKIPEDIA

Green's theorem

In mathematics, **Green's theorem** gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. It is named after George Green, though its first proof is due to Bernhard Riemann[1] and is the two-dimensional special case of the more general Kelvin–Stokes theorem.

Contents

Theorem

Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and let D be the region bounded by C. If L and M are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then[2][3]

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

where the path of integration along C is anticlockwise.

In physics, Green's theorem is mostly used to solve two-dimensional flow integrals, stating that the sum of fluid outflows from a volume is equal to the total outflow summed about an enclosing area. In plane geometry, and in particular, area surveying, Green's theorem can be used to determine the area and centroid of plane figures solely by integrating over the perimeter.

Proof when *D* is a simple region

The following is a proof of half of the theorem for the simplified area D, a type I region where C_1 and C_3 are curves connected by vertical lines (possibly of zero length). A similar proof exists for the other half of the theorem when D is a type II region where C_2 and C_4 are curves connected by horizontal lines (again, possibly of zero length). Putting these two parts together, the theorem is thus proven for regions of type III (defined as regions which are both type I and type II). The general case can then be deduced from this special case by decomposing D into a set of type III regions.

If it can be shown that

$$\oint_C L\,dx = \iint_D \left(-\frac{\partial L}{\partial y}\right) dA \qquad (1)$$

and

$$\oint_C M\,dy = \iint_D \left(\frac{\partial M}{\partial x}\right) dA \qquad (2)$$

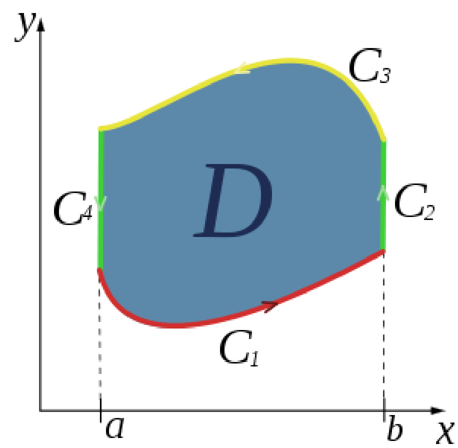
are true, then Green's theorem follows immediately for the region D. We can prove (1) easily for regions of type I, and (2) for regions of type II. Green's theorem then follows for regions of type III.

Assume region D is a type I region and can thus be characterized, as pictured on the right, by

$$D = \{(x,y) | a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous functions on $[a, b]$. Compute the double integral in (1):

$$\begin{aligned} \iint_D \frac{\partial L}{\partial y}\, dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial L}{\partial y}(x,y)\, dy\, dx \\ &= \int_a^b \Big\{ L(x, g_2(x)) - L(x, g_1(x)) \Big\}\, dx. \qquad (3) \end{aligned}$$



If D is a simple region with its boundary consisting of the curves C_1, C_2, C_3, C_4, half of Green's theorem can be demonstrated.

Now compute the line integral in (1). C can be rewritten as the union of four curves: C_1, C_2, C_3, C_4.

With C_1, use the parametric equations: $x = x$, $y = g_1(x)$, $a \le x \le b$. Then

$$\int_{C_1} L(x,y)\, dx = \int_a^b L(x, g_1(x))\, dx.$$

With C_3, use the parametric equations: $x = x$, $y = g_2(x)$, $a \le x \le b$. Then

$$\int_{C_3} L(x,y)\, dx = -\int_{-C_3} L(x,y)\, dx = -\int_a^b L(x, g_2(x))\, dx.$$

The integral over C_3 is negated because it goes in the negative direction from b to a, as C is oriented positively (anticlockwise). On C_2 and C_4, x remains constant, meaning

$$\int_{C_4} L(x,y)\, dx = \int_{C_2} L(x,y)\, dx = 0.$$

Therefore,

$$\begin{aligned} \int_C L\, dx &= \int_{C_1} L(x,y)\, dx + \int_{C_2} L(x,y)\, dx + \int_{C_3} L(x,y)\, dx + \int_{C_4} L(x,y)\, dx \\ &= -\int_a^b L(x, g_2(x))\, dx + \int_a^b L(x, g_1(x))\, dx. \qquad (4) \end{aligned}$$

Combining (3) with (4), we get (1) for regions of type I. A similar treatment yields (2) for regions of type II. Putting the two together, we get the result for regions of type III.

Proof for rectifiable Jordan curves

We are going to prove the following

Theorem. Let Γ be a rectifiable, positively oriented Jordan curve in $\mathbf{R}^2$ and let R denote its inner region. Suppose that $A, B : \overline{R} \longrightarrow \mathbf{R}$ are continuous functions with the property that A has second partial derivative at every point of R, B has first partial derivative at every point of R and that the functions $D_1 B$, $D_2 A : R \longrightarrow \mathbf{R}$ are Riemann-integrable over R. Then

$$\int_\Gamma A\, dx + B\, dy = \int_R \left(D_1 B(x,y) - D_2 A(x,y) \right)\, d(x,y).$$

We need the following lemmas:

Lemma 1 (Decomposition Lemma). Assume Γ is a rectifiable, positively oriented Jordan curve in the plane and let R be its inner region. For every positive real δ, let $\mathcal{F}(\delta)$ denote the collection of squares in the plane bounded by the lines $x = m\delta, y = m\delta$, where m runs through the set of integers. Then, for this δ, there exists a decomposition of $\overline{R}$ into a finite number of non-overlapping subregions in such a manner that

(i) Each one of the subregions contained in R, say $R_1, R_2, \cdots, R_k$, is a square from $\mathcal{F}(\delta)$.

(ii) Each one of the remaining subregions, say $R_{k+1}, \cdots, R_s$, has as boundary a rectifiable Jordan curve formed by a finite number of arcs of Γ and parts of the sides of some square from $\mathcal{F}(\delta)$.

(iii) Each one of the border regions $R_{k+1}, \cdots, R_s$ can be enclosed in a square of edge-length 2δ.

(iv) If Γ_i is the positively oriented boundary curve of R_i, then $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_s$.

(v) The number $s-k$ of border regions is no greater than $4\left(\dfrac{\Lambda}{\delta}+1\right)$, where Λ is the length of Γ.

Lemma 2. Let Γ be a rectifiable curve in the plane and let $\Delta_\Gamma(h)$ be the set of points in the plane whose disance from (the range of) Γ is at most h. The outer Jordan content of this set satisfies $\overline{c}\ \Delta_\Gamma(h) \leq 2h\Lambda + \pi h^2$.

Lemma 3. Let Γ be a rectifiable curve in $\mathbf{R}^2$ and let f **: range of** $\Gamma \longrightarrow \mathbf{R}$ be a continuous function. Then

$\left|\int_\Gamma f(x,y)\,dy\right|$ and $\left|\int_\Gamma f(x,y)\,dx\right|$ are $\leq \frac{1}{2}\Lambda\Omega_f$, where Ω_f is the oscillation of f on the range of Γ.

Now we are in position to prove the Theorem:

Proof of Theorem. Let ε be an arbitrary positive real number. By continuity of A, B and compactness of $\overline{R}$, given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that whenever two points of $\overline{R}$ are less than $2\sqrt{2}\,\delta$ apart, their images under A, B are less than ε apart. For this δ, consider the decomposition given by the previous Lemma. We have

$$\int_\Gamma A\,dx + B\,dy = \sum_{i=1}^{k}\int_{\Gamma_i} A\,dx + B\,dy \quad + \sum_{i=k+1}^{s}\int_{\Gamma_i} A\,dx + B\,dy.$$

Put $\varphi := D_1 B - D_2 A$.

For each $i \in \{1, \cdots, k\}$, the curve Γ_i is a positively oriented square, for which Green's formula holds. Hence

$$\sum_{i=1}^{k}\int_{\Gamma_i} A\,dx + B\,dy = \sum_{i=1}^{k}\int_{R_i} \varphi = \int_{\bigcup_{i=1}^{k} R_i} \varphi.$$

Every point of a border region is at a distance no greater than $2\sqrt{2}\,\delta$ from Γ. Thus, if K is the union of all border regions, then $K \subset \Delta_\Gamma(2\sqrt{2}\,\delta)$; hence $c(K) \leq \overline{c}\,\Delta_\Gamma(2\sqrt{2}\,\delta) \leq 4\sqrt{2}\,\delta + 8\pi\delta^2$, by Lemma 2. Notice that

$\int_R \varphi - \int_{\bigcup_{i=1}^{k} R_i} \varphi = \int_K \varphi$. This yields

$$\left|\sum_{i=1}^{k}\int_{\Gamma_i} A\,dx + B\,dy \quad - \int_R \varphi\right| \leq M\delta(1 + \pi\sqrt{2}\,\delta) \text{ for some } M > 0.$$

We may as well choose δ so that the RHS of the last inequality is $< \varepsilon$.

The remark in the beginning of this proof implies that the oscillations of A and B on every border region is at most ε. We have

$$\left|\sum_{i=k+1}^{s}\int_{\Gamma_i} A\,dx + B\,dy\right| \leq \frac{1}{2}\varepsilon\sum_{i=k+1}^{s}\Lambda_i.$$

By Lemma 1(iii),

$$\sum_{i=k+1}^{s}\Lambda_i \leq \Lambda + (4\delta)\,4\left(\frac{\Lambda}{\delta}+1\right) \leq 17\Lambda + 16.$$

Combining these, we finally get

$$\left|\int_\Gamma A\,dx + B\,dy \quad - \int_R \varphi\right| < C\varepsilon,$$

for some $C > 0$. Since this is true for every $\varepsilon > 0$, we are done.

Validity under different hypothesis

The hypothesis of the last theorem are not the only ones under which Green's formula is true. Another common set of conditions is the following:

The functions $A, B : \overline{R} \longrightarrow \mathbf{R}$ are still assumed to be continuous. However, we now require them to be Fréchet-differentiable at every point of R. This implies the existence of all directional derivatives, in particular $D_{e_i}A =: D_i A, D_{e_i}B =: D_i B$, $i = 1, 2$, where, as usual, (e_1, e_2) is the canonical ordered basis of $\mathbf{R}^2$. In addition, we require the function $D_1 B - D_2 A$ to be Riemann-integrable over R.

It suffices to prove this for squares which are contained in R and have sides parallel to the axes. The proof then follows the lines of the method employed to prove the Cauchy-Goursat Theorem for triangles.

As a corollary of this, we get the Cauchy Integral Theorem for rectifiable Jordan curves:

Theorem (Cauchy). If Γ is a rectifiable Jordan curve in $\mathbf{C}$ and if $f:$ **closure of inner region of** $\Gamma \longrightarrow \mathbf{C}$ is a continuous mapping holomorphic throughout the inner region of Γ, then

$$\int_\Gamma f = 0,$$

the integral being a complex contour integral.

Proof. We regard the complex plane as $\mathbf{R}^2$. Now, define $u, v : \overline{R} \longrightarrow \mathbf{R}$ to be such that $f(x+iy) = u(x,y) + iv(x,y)$. These functions are clearly continuous. It is well-known that u and v are Fréchet-differentiable and that they satisfy the Cauchy-Riemann equations: $D_1 v + D_2 u = D_1 u - D_2 v = \textbf{zero function}$.

Now, analysing the sums used to define the complex contour integral in question, it is easy to realize that

$$\int_\Gamma f = \int_\Gamma u\,dx - v\,dy \quad + i \int_\Gamma v\,dx + u\,dy,$$

the integrals on the RHS being usual line integrals. These remarks allow us to aply Green's Theorem to each one of these line integrals, finishing the proof.

Measure-theoretic assumptions

Green's formula also holds when, besides continuity assumptions,

(i) The functions $D_i A, D_i B,\ i = 1, 2$, are defined at every point of R, with the exception of a countable subset.

(ii) The function $D_1 B - D_2 A$ is Lebesgue-integrable over R.

Multiply-connected Regions

Theorem. Let $\Gamma_0, \Gamma_1, \cdots, \Gamma_n$ be positively oriented rectifiable Jordan curves in $\mathbf{R}^2$ satisfying

$$\begin{aligned} \Gamma_i &\subset R_0, && \text{if } 1 \le i \le n \\ \Gamma_i &\subset \mathbf{R}^2 \setminus \overline{R}_j, && \text{if } 1 \le i, j \le n \text{ and } i \ne j, \end{aligned}$$

where R_i is the inner region of Γ_i. Let

$$D = R_0 \setminus (\overline{R}_1 \cup \overline{R}_2 \cup \cdots \cup \overline{R}_n).$$

Suppose $p : \overline{D} \longrightarrow \mathbf{R}$ and $q : \overline{D} \longrightarrow \mathbf{R}$ are continuous functions whose restriction to D is Fréchet-differentiable. If the function

$$(x,y) \longmapsto \frac{\partial q}{\partial e_1}(x,y) - \frac{\partial p}{\partial e_2}(x,y)$$

is Riemann-integrable over D, then

$$\int_{\Gamma_0} p(x,y)\,dx + q(x,y)\,dy - \sum_{i=1}^{n} \int_{\Gamma_i} p(x,y)\,dx + q(x,y)\,dy = \int_D \left\{ \frac{\partial q}{\partial e_1}(x,y) - \frac{\partial p}{\partial e_2}(x,y) \right\} d(x,y).$$

Relationship to Stokes' theorem

Green's theorem is a special case of the Kelvin–Stokes theorem, when applied to a region in the xy-plane:

We can augment the two-dimensional field into a three-dimensional field with a z component that is always 0. Write $\mathbf{F}$ for the vector-valued function $\mathbf{F} = (L, M, 0)$. Start with the left side of Green's theorem:

$$\oint_C (L\,dx + M\,dy) = \oint_C (L, M, 0) \cdot (dx, dy, dz) = \oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Kelvin–Stokes Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS.$$

The surface S is just the region in the plane D, with the unit normals $\hat{\mathbf{n}}$ pointing up (in the positive z direction) to match the "positive orientation" definitions for both theorems.

The expression inside the integral becomes

$$\nabla \times \mathbf{F} \cdot \hat{\mathbf{n}} = \left[\left(\frac{\partial 0}{\partial y} - \frac{\partial M}{\partial z}\right)\mathbf{i} + \left(\frac{\partial L}{\partial z} - \frac{\partial 0}{\partial x}\right)\mathbf{j} + \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathbf{k}\right] \cdot \mathbf{k} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right).$$

Thus we get the right side of Green's theorem

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\, dA.$$

Green's theorem is also a straightforward result of the general Stokes' theorem using differential forms and exterior derivatives:

$$\oint_C L\,dx + M\,dy = \oint_{\partial D} \omega = \int_D d\omega = \int_D \frac{\partial L}{\partial y}\, dy \wedge\, dx + \frac{\partial M}{\partial x}\, dx \wedge\, dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\, dx\, dy.$$

Relationship to the divergence theorem

Considering only two-dimensional vector fields, Green's theorem is equivalent to the two-dimensional version of the divergence theorem:

$$\iiint_V (\nabla \cdot \mathbf{F})\ dV = \oiint_{\partial V} (\mathbf{F} \cdot \mathbf{n})\, dS.$$

where $\nabla \cdot \mathbf{F}$ is the divergence on the two-dimensional vector field $\mathbf{F}$, and $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector on the boundary.

To see this, consider the unit normal $\hat{\mathbf{n}}$ in the right side of the equation. Since in Green's theorem $d\mathbf{r} = (dx, dy)$ is a vector pointing tangential along the curve, and the curve C is the positively oriented (i.e. anticlockwise) curve along the boundary, an outward normal would be a vector which points 90° to the right of this; one choice would be $(dy, -dx)$. The length of this vector is $\sqrt{dx^2 + dy^2} = ds.$ So $(dy, -dx) = \hat{\mathbf{n}}\, ds.$

Start with the left side of Green's theorem:

$$\oint_C (L\,dx + M\,dy) = \oint_C (M, -L) \cdot (dy, -dx) = \oint_C (M, -L) \cdot \hat{\mathbf{n}}\, ds.$$

Applying the two-dimensional divergence theorem with $\mathbf{F} = (M, -L)$, we get the right side of Green's theorem:

$$\oint_C (M, -L) \cdot \hat{\mathbf{n}}\, ds = \iint_D (\nabla \cdot (M, -L))\, dA = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\, dA.$$

Area calculation

Green's theorem can be used to compute area by line integral.[4] The area of D is given by $A = \iint_D dA$. Then if we choose L and M such that $\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = 1$, the area is given by $A = \oint_C (L\,dx + M\,dy)$.

Possible formulas for the area of D include:[4] $A = \oint_C x\,dy = -\oint_C y\,dx = \tfrac{1}{2}\oint_C (-y\,dx + x\,dy).$

See also

- Planimeter
- Method of image charges – A method used in electrostatics that takes advantage of the uniqueness theorem (derived from Green's theorem)
- Shoelace formula – A special case of Green's theorem for simple polygons

References

1. George Green, *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* (Nottingham, England: T. Wheelhouse, 1828). Green did not actually derive the form of "Green's theorem" which appears in this article; rather, he derived a form of the "divergence theorem", which appears on pages 10-12 (https://books.google.com/books?id=GwYXAAAAYAAJ&pg=PA10#v=onepage&q&f=false) of his *Essay*.
   In 1846, the form of "Green's theorem" which appears in this article was first published, without proof, in an article by Augustin Cauchy: A. Cauchy (1846) "Sur les intégrales qui s'étendent à tous les points d'une courbe fermée" (https://archive.org/stream/ComptesRendusAcademieDesSciences0023/ComptesRendusAcadmieDesSciences-Tome023-Juillet-dcembre1846#page/n254/mode/1up) (On integrals that extend over all of the points of a closed curve), *Comptes rendus*, **23**: 251-255. (The equation appears at the bottom of page 254, where (S) denotes the line integral of a function *k* along the curve *s* that encloses the area S.)
   A proof of the theorem was finally provided in 1851 by Bernhard Riemann in his inaugural dissertation: Bernhard Riemann (1851) *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (https://books.google.com/books?id=PpALAAAAYAAJ&pg=PP5#v=onepage&q&f=false) (Basis for a general theory of functions of a variable complex quantity), (Göttingen, (Germany): Adalbert Rente, 1867); see pages 8 - 9.

2. Mathematical methods for physics and engineering, K.F. Riley, M.P. Hobson, S.J. Bence, Cambridge University Press, 2010, ISBN 978-0-521-86153-3
3. Vector Analysis (2nd Edition), M.R. Spiegel, S. Lipschutz, D. Spellman, Schaum's Outlines, McGraw Hill (USA), 2009, ISBN 978-0-07-161545-7
4. Stewart, James. *Calculus* (6th ed.). Thomson, Brooks/Cole.

Further reading

- Ayres, F.; Mendelson, E. (2009). *Calculus*. Schaum's Outline (5th ed.). ISBN 978-0-07-150861-2.
- Wrede, R.; Spiegel, M. R. (2010). *Advanced Calculus*. Schaum's Outline (3rd ed.). ISBN 978-0-07-162366-7.

External links

- Green's Theorem on MathWorld (http://mathworld.wolfram.com/GreensTheorem.html)