Finite Difference Schemes

Computational Fluid Dynamics SG2212 (20100123)

1 Finite differences for the integration of ODEs

Ordinary differential equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u,t),\ u^n = u(t^n),\ f^n = (u^n, t^n),\ t^n = n\Delta t \tag{1}$$

Explicit Euler scheme, order $\mathcal{O}(\Delta t)$:

$$u^{n+1} = u^n + \Delta t \cdot f^n \tag{2}$$

Implicit Euler scheme, order $\mathcal{O}(\Delta t)$:

$$u^{n+1} = u^n + \Delta t \cdot f^{n+1} \tag{3}$$

(Generalised) Crank-Nicolson scheme:

$$u^{n+1} = u^n + \Delta t([1-\theta]\cdot f^n + [\theta]\cdot f^{n+1})\ ,\quad 0 \le \theta \le 1 \tag{4}$$

The standard Crank-Nicolson scheme is given by $\theta = 0.5$ with order $\mathcal{O}(\Delta t^2)$; the explicit and implict Euler schemes are obtained with $\theta = 0$ and $\theta = 1$, respectively.

Standard Runge-Kutta scheme (RK4), order $\mathcal{O}(\Delta t^4)$:

$$u^{n+1} = u^n + \frac{\Delta t}{6}(f^n + 2k_1 + 2k_2 + k_3) \tag{5}$$

$$\begin{aligned}
\text{with}:\ u_1 &= u^n + \frac{\Delta t}{2} f^n,\ k_1 = f(u_1, t^{n+\frac{1}{2}}),\ t^{n+\frac{1}{2}} = t^n + \frac{\Delta t}{2} && (6)\\
u_2 &= u^n + \frac{\Delta t}{2} k_1,\ k_2 = f(u_2, t^{n+\frac{1}{2}}) && (7)\\
u_3 &= u^n + \Delta t k_2,\ k_3 = f(u_3, t^{n+1}) && (8)
\end{aligned}$$

2 Finite difference formulas for first derivatives

Left-sided finite differece scheme first order:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_i - u_{i-1}}{\Delta x} + \frac{\Delta x}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \dots \tag{9}$$

Left-sided finite differece scheme second order:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} + \frac{\Delta x^2}{3}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \dots \tag{10}$$

Right-sided finite differece scheme first order:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_{i+1} - u_i}{\Delta x} - \frac{\Delta x}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \dots \tag{11}$$

Right-sided finite differece scheme second order:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2\Delta x} + \frac{\Delta x^2}{3}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \dots \tag{12}$$

Central finite differece scheme second order:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{u_{i+1} - u_{i-1}}{2\Delta x} - \frac{\Delta x^2}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \ldots \tag{13}$$

Central finite differece scheme fourth order:

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} + \frac{\Delta x^4}{30}\left.\frac{\partial^5 u}{\partial x^5}\right|_{x_i} + \ldots \tag{14}$$

3 Finite difference formulas for second derivatives

Left-sided finite differece scheme first order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{u_i - 2u_{i-1} + u_{i-2}}{\Delta x^2} + \Delta x\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \ldots \tag{15}$$

Left-sided finite differece scheme second order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{2u_i - 5u_{i-1} + 4u_{i-2} - u_{i-3}}{\Delta x^2} - \frac{11\Delta x^2}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} + \ldots \tag{16}$$

Right-sided finite differece scheme first order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{u_{i+2} - 2u_{i+1} + u_i}{\Delta x^2} - \Delta x\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} + \ldots \tag{17}$$

Right-sided finite differece scheme second order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{2u_i - 5u_{i+1} + 4u_{i+2} - u_{i+3}}{\Delta x^2} + \frac{11\Delta x^2}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} + \ldots \tag{18}$$

Central finite differece scheme second order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} - \frac{\Delta x^2}{12}\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} + \ldots \tag{19}$$

Central finite differece scheme fourth order:

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} = \frac{-u_{i+2} + 16u_{i+1} - 30u_i + 16u_{i-1} - u_{i-2}}{12\Delta x^2} + \frac{\Delta x^4}{90}\left.\frac{\partial^6 u}{\partial x^6}\right|_{x_i} + \ldots \tag{20}$$

4 Finite difference formulas for third derivatives

Central finite differece scheme second order:

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} = \frac{u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}}{2\Delta x^3} - \frac{\Delta x^2}{4}\left.\frac{\partial^5 u}{\partial x^5}\right|_{x_i} + \ldots \tag{21}$$

Central finite differece scheme fourth order:

$$\left.\frac{\partial^3 u}{\partial x^3}\right|_{x_i} = \frac{-u_{i+3} + 8u_{i+2} - 13u_{i+1} + 13u_{i-1} - 8u_{i-2} + u_{i-3}}{8\Delta x^3} + \frac{7\Delta x^4}{120}\left.\frac{\partial^7 u}{\partial x^7}\right|_{x_i} + \ldots \tag{22}$$

5 Finite difference formulas for fourth derivatives

Central finite differece scheme second order:

$$\left.\frac{\partial^4 u}{\partial x^4}\right|_{x_i} = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{\Delta x^4} - \frac{\Delta x^2}{6}\left.\frac{\partial^6 u}{\partial x^6}\right|_{x_i} + \ldots \tag{23}$$