Notes for:
Turbulence Modelling
(in 5C1218 and 5C5112)

Stefan Wallin
Dept. of Mechanics, KTH and Systems Dept, FOI
stefan.wallin@foi.se
March 8, 2006

Outline

- Introduction and background
- One-equation models
- Two-equation models
- Reynolds stress models

Introduction

Turbulent flow governed by Navier-Stokes equation. Why modelling?

- DNS: Direct Numerical Simulation
  - Solution of the 3D time dependent Navier-Stokes equations
  - Computational effort scales rapidly with Reynolds number
  - Used for low Reynolds number generic flows
  - Gives detailed knowledge about turbulence
  - Not (yet?) practically useful for “CFD applications”

- Computational effort for wall bounded turbulence
  - Number of grid points

$$N_{\mathrm{nodes}} \sim \frac{L^2\delta}{l_*^3} \sim \frac{L^3(\delta/L)}{(\nu/u_\tau)^3} \sim \left(\frac{LU}{\nu}\right)^3\left(\frac{\delta}{L}\right)\left(\frac{u_\tau}{U}\right)^3 \sim Re_L^3 Re_L^{-1/5} Re_L^{-3/10} \sim Re_L^{5/2}$$

  - Number of time steps

$$N_{\Delta t} \sim \frac{T}{t_*} \sim \frac{L/U}{\nu/u_\tau^2} \sim \frac{LU}{\nu}\left(\frac{u_\tau}{U}\right)^2 \sim Re_L Re_L^{-1/5} \sim Re_L^{4/5}$$

  - Computational effort

$$CPU \sim N_{\mathrm{nodes}} N_{\Delta t} \sim Re_L^{33/10} \sim Re_L^{3.3}$$

  - $Re_L \sim 10^4$ can be simulated ($N_{\mathrm{nodes}} \sim 10^8$, one CPU-year on a PC).

    Full scale airplane with $Re_L \sim 10^8$ requires $N_{\mathrm{nodes}} \sim 10^{18}$ and 10^{13} times more CPU. Can be done in year 2070 if the computers continue to develop according to Moorse’s law (doubled every 18 month).

Thus, turbulent flows need to be modelled.

Turbulence modelling

Reynolds averaged Navier-Stokes equations (RANS)

- Navier-Stokes equations ($\tilde{u}, \tilde{p}$ denote instantaneous values)
  - Conservation of mass

$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0 \tag{1}$$

  - Conservation of momentum

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_k \frac{\partial \tilde{u}_i}{\partial x_k} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \tilde{u}_i}{\partial x_k}\right) \tag{2}$$

  - Conservation of a passive scalar quantity $\tilde{\theta}$

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u}_k \frac{\partial \tilde{\theta}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(D \frac{\partial \tilde{\theta}}{\partial x_k}\right) \tag{3}$$

- Reynolds decomposition

$$\tilde{u}_i(\boldsymbol{x}, t) = U_i(\boldsymbol{x}) + u_i(\boldsymbol{x}, t)$$

where $U_i(\boldsymbol{x}) = \overline{\tilde{u}_i(\boldsymbol{x}, t)}$ and $\overline{u_i(\boldsymbol{x}, t)} = 0$

- The “mean” is time average, ensemble average or averaging in homogeneous directions. $U_i(\boldsymbol{x})$ may actually vary in time ($U_i(\boldsymbol{x}, t)$) with a time scale much longer than the turbulent time scale.
- Take the mean of the Navier-Stokes equations -> RANS

$$\frac{\partial U_i}{\partial x_i} = 0 \tag{4}$$

$$\frac{DU_i}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial U_i}{\partial x_k} - \overline{u_i u_k}\right) \tag{5}$$

$$\frac{D\Theta}{Dt} = \frac{\partial}{\partial x_k}\left(D \frac{\partial \Theta}{\partial x_k} - \overline{u_k \theta}\right) \tag{6}$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + U_k \frac{\partial}{\partial x_k}$

- Reynolds stresses, $\overline{u_i u_j}$ (and scalar flux vector $\overline{u_i \theta}$)
  - Appears because of the non-linear term
  - Not “small”
  - Significant effects on the flow
  - Needs to be modelled in terms of mean flow quantities
  - Reduces the problem to steady (or slowly varying)
  - 2D assumptions possible

The aim is to construct a closed system of equations for the one-point quantities (U_i, P, $\overline{u_i u_j}$).

- Only one governing lengthscale
- Spectra are assumed to be self-similar

Equation for the Reynolds stress tensor, $\overline{u_i u_j}$ (and for $\overline{u_i \theta}$)

- Can be derived from the Navier-Stokes equations
- Contains, however, higher order moments like $\overline{u_i u_j u_k}$
- Equations for $\overline{u_i u_j u_k}$ contain even higher moments: the closure problem
- Modelling needed at some stage.

Equation for the Reynolds stress tensor, $\overline{u_i u_j}$

Equation for the fluctuating velocity u_i is derived by subtracting the RANS equation (5) from the Navier-Stokes equation (2)

$$\frac{\partial u_i}{\partial t} = \frac{\partial(U_i + u_i)}{\partial t} - \frac{\partial U_i}{\partial t} = \dots \tag{7}$$

Then, the equation for $u_i u_j$ is derived by the following

$$\frac{\partial u_i u_j}{\partial t} = u_i \frac{\partial u_j}{\partial t} + u_j \frac{\partial u_i}{\partial t} = \dots \tag{8}$$

The equation for $\overline{u_i u_j}$ is then derived by taking the mean of (8)

$$\frac{D\overline{u_i u_j}}{Dt} = \mathcal{P}_{ij} - \varepsilon_{ij} + \Pi_{ij} + \mathcal{D}_{ij} \tag{9}$$

where

- Production, transfers energi from mean flow to fluctuations

$$\mathcal{P}_{ij} = -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} \tag{10}$$

- Dissipation, transfers energy from fluctuations to heat

$$\varepsilon_{ij} = 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{11}$$

- Pressure-strain, redistributes energy among the components ($\Pi_{ii} = 0$)

$$\Pi_{ij} = \overline{\frac{1}{\rho}p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{12}$$

- Diffusion, redistributes fluctuations in space

$$\mathcal{D}_{ij} = -\frac{\partial}{\partial x_k}\left[\overline{u_i u_j u_k} + \frac{1}{\rho}(\overline{u_i p}\delta_{jk} + \overline{u_j p}\delta_{ik}) - \nu\frac{\partial \overline{u_i u_j}}{\partial x_k}\right] \tag{13}$$

The equation for the turbulent kinetic energy, $K \equiv \overline{u_i u_i}/2$, is derived by taking half of the trace of the equation (9)

$$\frac{DK}{Dt} = \mathcal{P} - \varepsilon + \mathcal{D} \tag{14}$$

where

$$\mathcal{P} = -\overline{u_i u_k}\frac{\partial U_i}{\partial x_k} \tag{15}$$

$$\varepsilon = \nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} \tag{16}$$

$$\mathcal{D} = -\frac{\partial}{\partial x_k}\left[\frac{1}{2}\overline{u_i u_i u_k} + \frac{1}{\rho}\overline{u_k p} - \nu\frac{\partial K}{\partial x_k}\right] \tag{17}$$

Example: Homogeneous shear flow

$$\frac{\partial U}{\partial y} = S, \; V = W = 0, \; \mathcal{D}_{ij} = \mathcal{D} = 0$$

Gives equation for K

$$\frac{\partial K}{\partial t} = -\overline{uv}S - \varepsilon$$

and equations for the Reynolds stresses

$$\frac{\partial \overline{u^2}}{\partial t} = -2\overline{uv}S + \Pi_{11} - \varepsilon_{11}$$

$$\frac{\partial \overline{v^2}}{\partial t} = \Pi_{22} - \varepsilon_{22}$$

$$\frac{\partial \overline{w^2}}{\partial t} = \Pi_{33} - \varepsilon_{33}$$

$$\frac{\partial \overline{uv}}{\partial t} = -\overline{v^2}S + \Pi_{12} - \varepsilon_{12}$$

Note: all energy production in the $\overline{u^2}$ component and redistributed to the other components by Π_{ij} to an asymptotic state (Tavoularis & Corrsin, 1981) where

$$\frac{\overline{u^2}}{2K} = 0.53 \quad \frac{\overline{v^2}}{2K} = 0.19 \quad \frac{\overline{w^2}}{2K} = 0.28 \quad \frac{\overline{uv}}{2K} = -0.15 \quad \frac{SK}{\varepsilon} = 6$$

which give

$$\frac{\mathcal{P}}{\varepsilon} = \frac{-\overline{uv}S}{\varepsilon} = -2\frac{\overline{uv}}{2K}\frac{SK}{\varepsilon} = 2 \times 0.15 \times 6 = 1.8$$

Eddy-viscosity models (EVM)

Boussinesq (1877) made the assumption

$$\overline{uv} = -\nu_T \frac{\partial U}{\partial y}$$

This was later generalized to the eddy-viscosity, or Boussinesq, assumption

$$\overline{u_i u_j} - \frac{2}{3} K \delta_{ij} = -2\nu_T S_{ij} \tag{18}$$

where the symmetric and antisymmetric parts of the velocity gradient are

$$S_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \; \Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

and the Reynolds stress anisotropy

$$a_{ij} \equiv \frac{\overline{u_i u_j}}{K} - \frac{2}{3}\delta_{ij} \tag{19}$$

The eddy-viscosity assumption may be written

$$a_{ij} = -\frac{2\nu_T}{K} S_{ij} \tag{20}$$

The scalar flux is modelled accordingly

$$\overline{u_i \theta} = -D_T \frac{\partial \Theta}{\partial x_i} \tag{21}$$

where the eddy diffusivity D_T may be related to the eddy viscosity by a turbulent Schmidt number (of the order of unity).

$$D_T = \frac{\nu_T}{\sigma_T} \tag{22}$$

The RANS equation (5) can now be rewritten as

$$\frac{DU_i}{Dt} = -\frac{\partial}{\partial x_i}\left(\frac{P}{\rho} + \frac{2}{3}K\right) + \frac{\partial}{\partial x_k}[2(\nu + \nu_T) S_{ij}] \tag{23}$$

where the turbulent kinetic energy, K, may be absorbed into an effective pressure.

The following observations can be made:

- Zero strain ($S_{ij} = 0$) gives zero anisotropy ($a_{ij} = 0$). History effects are not well described.
- S_{ij} and a_{ij} are not in general aligned and the deviation is large. Thus, the eddy-viscosity relation is only valid for one component in thin shear layers (as originally proposed by Boussinesq). This is also the case for the scalar flux that is not aligned with the gradient.
- S_{ij} is invariant of rotation, which will only enter into Ω_{ij}. Since a_{ij} depends on S_{ij} but not on Ω_{ij} no rotational effects will enter into a_{ij}. Also the K equation is unaffected by rotation. So, rotation effects are not well described by eddy-viscosity models.

The eddy viscosity is related to the large scale (most energetic) turbulent scales as

$$\nu_T \sim V \cdot L \tag{24}$$

where V and L are the velocity and length scales respectivelly.

Classification of eddy-viscosity models:

- Algebraic models or zero-equation models:
  - V and L related to mean flow field and global geometry (wall distance, wake thickness, et.c.)
  - Works well for attached boundary layers and other thin shear layers
  - Not very general
- One-equation models:
  - One transport equation for K or ν_T.
  - Additional information from global conditions (typically wall distance)
  - Works well for attached boundary layers and other thin shear layers
  - Not very general, but more than algebraic models
  - Example: Spalart-Allmaras (1992) (a reasonable and robust model for external aerodynamics)

- Two-equation models:
  - Two transport equations for the turbulence scales, typically $K-\varepsilon$ or $K-\omega$.
  - Completely determined in terms of local quantities (except near-wall corrections which may be dependent on wall distance)
  - Works well for attached boundary layers
  - Somewhat more general than algebraic and one-equation models.
  - Model transport equations loosely connected to the exact equations.
  - Examples:
    Standard $K-\varepsilon$ model (Launder & Spalding 1974)
    Wilcox (1988, ...) $K-\omega$ models
    Menter (1994) SST $K-\omega$ model (performing reasonable well also in separated flows)

Zero-equation models

Constant turbulent viscosity, ν_T (and diffusivity D_T)

- Fair assumption only in free shear flows
- elsewhere, no general constant ν_T can be found.

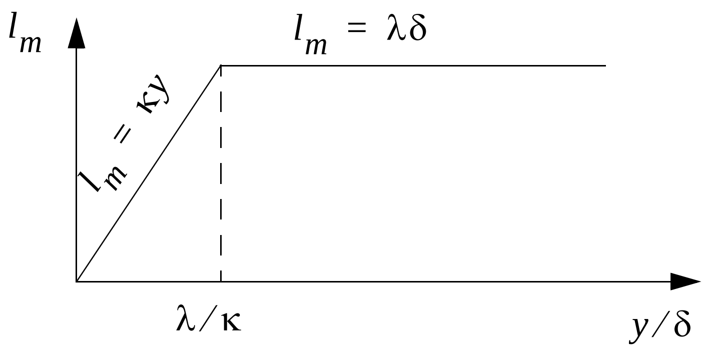
Mixing length models

$$V \sim l_m \left| \frac{\partial U}{\partial y} \right| \text{ and } L \sim l_m$$

which gives:

$$\nu_T = l_m^2 \left| \frac{\partial U}{\partial y} \right| \text{ or } -\overline{uv} = l_m^2 \left| \frac{\partial U}{\partial y} \right| \frac{\partial U}{\partial y} \qquad (25)$$

- Free shear flows: Constant l_m/δ is a common choice. Ex:
  - plane jet: $l_m/\delta \approx 0.9$
  - round jet: $l_m/\delta \approx 0.075$
  - plane wake: $l_m/\delta \approx 0.16$
- Wall-bounded flows: l_m chosen as a ramp function with $\kappa = 0.4$ and $\lambda = 0.1$



- Lack of generality!
- Example:
  - Baldwin-Lomax (1978) (rather accurate for attached boundary layers)

One-equation models

The basic idea is to decouple the turbulent velocity scale, V, from the mean strain field and determine it from a transport equation. If the transport equation for the turbulent kinetic energy, K, is used the velocity scale may be derived from that as $V \sim \sqrt{K}$ (Kolmogorov & Prandtl)

$$\nu_T = c'_\mu \sqrt{K} L \tag{26}$$

where c'_μ is an empirical constant and the equation for K (14) is

$$\frac{DK}{Dt} = \mathcal{P} - \varepsilon + \mathcal{D} \tag{27}$$

where the terms are modelled as

$$\mathcal{P} = 2\nu_T S_{ij} S_{ij}$$

$$\varepsilon = c_D \frac{K^{3/2}}{L}$$

$$\mathcal{D} = \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_k}\right)\frac{\partial K}{\partial x_k}\right]$$

with the model coefficients $\sigma_k = 1.0$ and $c'_\mu c_D = 0.09$

The length scale, L, needs to be determined. This is a problem, as with zero-equation models.

Assume local equilibrium $\mathcal{P} \approx \varepsilon$ and thin shear layers:

$$\mathcal{P} = 2\nu_T S_{ij} S_{ij} = \nu_T \left(\frac{\partial U}{\partial y}\right)^2$$

$$\varepsilon = c_D \frac{K^{3/2}}{L} = c_D \frac{(\sqrt{K}L)^3}{L^4} = c_D \frac{\nu_T^3}{L^4 c'^3_\mu}$$

$$\mathcal{P} \approx \varepsilon \Rightarrow \left(\frac{\partial U}{\partial y}\right)^2 = \frac{c_D \nu_T^2}{L^4 c'^3_\mu} \Rightarrow \nu_T = \sqrt{\frac{c'^3_\mu}{c_D}} L^2 \left|\frac{\partial U}{\partial y}\right|$$

One-equation models when $\mathcal{P} \approx \varepsilon$ is a mixing length model with

$$l_m = \left(\frac{c'^3_\mu}{c_D}\right)^{1/4} L$$

Or, l_m hypothesis only suitable when $\mathcal{P} \approx \varepsilon$. Fair in boundary layers.

Spalart & Allmaras one-equation model.

One attempt to try to avoid the problem with the length scale is to form a transport equation for ν_T directly. This has been proposed by Spalart & Allmaras (1994). The model reads (somewhat simplified)

$$\frac{D\nu_T}{Dt} = \mathcal{P}_{\nu_T} - \varepsilon_{\nu_T} + \mathcal{D}_{\nu_T} \tag{28}$$

The production of ν_T is modelled as

$$\mathcal{P}_{\nu_T} = c_{b1} S \nu_T$$

where S is the absolute value of the vorticity (reduces to $|\partial U / \partial y|$ in thin shear layers). c_{b1} is a model constant.

The dissipation of ν_T is modelled as

$$\varepsilon_{\nu_T} = c_{w1} f_w \left(\frac{\nu_T}{d}\right)^2$$

where c_{w1} is a constant and f_w is a complex function taking care of the outer part of the boundary layer. Here, a length scale is actually introduced with d, which is the wall distance.

The Spalart & Allmaras model consists of a lot of different empirical corrections and has been carefully calibrated for different flows and performs well in many different cases.

Two-equation models (eddy viscosity)

- Eddy-viscosity assumption

$$a_{ij} = -\frac{2\nu_T}{K}S_{ij} \tag{29}$$

Remember:

$$a_{ij} \equiv \frac{\overline{u_i u_j}}{K} - \frac{2}{3}\delta_{ij}, \; S_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \; \Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

- Model equations (standard $K - \varepsilon$ model):

$$\begin{aligned} \frac{DK}{Dt} &= \mathcal{P} - \varepsilon + \mathcal{D} \\ \frac{D\varepsilon}{Dt} &= \mathcal{P}_\varepsilon - \varepsilon_\varepsilon + \mathcal{D}_\varepsilon \end{aligned} \tag{30}$$

Modelling and calibration of the different terms

- The eddy-viscosity relation:

The standard form is derived from the natural velocity and length scales, $V \sim \sqrt{K}$ and $L \sim K^{3/2}/\varepsilon$, which gives

$$\nu_T = C_\mu \frac{K^2}{\varepsilon} \tag{31}$$

In thin shear flows $\mathcal{P} \approx \varepsilon$, which gives

$$\frac{\mathcal{P}}{\varepsilon} = 1 = \frac{-\overline{uv}\frac{\partial U}{\partial y}}{\varepsilon} = -a_{12}\frac{K}{\varepsilon}\frac{\partial U}{\partial y} \Rightarrow \frac{K}{\varepsilon}\frac{\partial U}{\partial y} = \frac{1}{-a_{12}}$$

used in the eddy-viscosity relations (29, 31)

$$-a_{12} = \frac{\nu_T}{K}\frac{\partial U}{\partial y} = C_\mu \frac{K}{\varepsilon}\frac{\partial U}{\partial y} = \frac{C_\mu}{-a_{12}} \Rightarrow C_\mu = a_{12}^2$$

In thin shear layers (e.g. boundary layers) $-a_{12} \approx 0.3$ (Bradshaw hypothesis), which gives $C_\mu = 0.09$

- Production of K, $\mathcal{P}$:

The exact form reduces to the modelled by applying the eddy-viscosity relation (29)

$$\text{exact: } \mathcal{P} = -\overline{u_i u_k}\frac{\partial U_i}{\partial x_k}, \text{ modelled: } \mathcal{P} = 2\nu_T S_{ij} S_{ji}$$

- Destruction terms ε and ε_ε:

ε in the K equation is modelled as a transport equation. The equation for ε is derived from the Navier-Stokes equation and the production, destruction and diffusion terms all need modelling.

The destruction term in the ε equation, ε_ε, is modelled by considering homogeneous decaying turbulence

$$\frac{dK}{dt} = -\varepsilon, \ \frac{d\varepsilon}{dt} = -\varepsilon_\varepsilon$$

The assumption is that the decay rates of K and ε should be self similar and differ by a constant factor

$$\frac{\frac{d\varepsilon}{dt}/\varepsilon}{\frac{dK}{dt}/K} = C_{\varepsilon 2} \Rightarrow \varepsilon_\varepsilon = C_{\varepsilon 2}\frac{\varepsilon^2}{K}$$

The evolution of K can then be derived as

$$\frac{K(t)}{K_0} = \left(1 + (C_{\varepsilon 2} - 1)\frac{\varepsilon_0}{K_0}t\right)^{-n}, \ n \equiv \frac{1}{C_{\varepsilon 2} - 1}$$

and $C_{\varepsilon 2}$ is calibrated from experiments. $C_{\varepsilon 2}$ can also be derived from theoretical analyses of the energy spectra. “Standard” value $C_{\varepsilon 2} = 1.92$.

- Production of ε, $\mathcal{P}_\varepsilon$:

Mainly dimensional arguments (weak coupling to the exact term) gives

$$\mathcal{P}_\varepsilon = C_{\varepsilon 1}\frac{\varepsilon}{K}\mathcal{P}$$

$C_{\varepsilon 1}$ calibrated from homogeneous ($\mathcal{D} = \mathcal{D}_\varepsilon = 0$) shear flow

$$\begin{aligned} \frac{dK}{dt} &= \mathcal{P} - \varepsilon \\ \frac{D\varepsilon}{Dt} &= C_{\varepsilon 1}\frac{\varepsilon}{K}\mathcal{P} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} \end{aligned} \qquad (32)$$

For long times K/ε approaches a constant, thus, from (32) it can be derived that

$$\frac{d}{dt}\left(\frac{K}{\varepsilon}\right) = \frac{1}{\varepsilon}\frac{dK}{dt} - \frac{K}{\varepsilon^2}\frac{D\varepsilon}{Dt} = -(C_{\varepsilon 1} - 1)\frac{\mathcal{P}}{\varepsilon} + (C_{\varepsilon 2} - 1) = 0$$

and from experiment (Tavoularis & Corrsin, 1981) $\mathcal{P}/\varepsilon \approx 1.8$. With $C_{\varepsilon 2} = 1.92$ one can derive $C_{\varepsilon 1} \approx 1.5$. “Standard” value $C_{\varepsilon 1} = 1.44$.

- Diffusion of K and ε, $\mathcal{D}$ and $\mathcal{D}_\varepsilon$

The simplest assumption is to assume gradient diffusion

$$\mathcal{D} = \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_k}\right], \ \mathcal{D}_\varepsilon = \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_k}\right]$$

The viscous diffusion is present also in the exact equations. The turbulent diffusion has to be modelled, and the gradient diffusion is not a very good approximation, but has been found to be a reasonable approximaion in thin shear flows. This will be discussed lated.

The Schmidt number in the K equation should be close to unity, $\sigma_K = 1.0$.

The Schmidt number in the ε equation, σ_ε, is calibrated from the log-layer in boundary layers. The log-law

$$\frac{U}{u_\tau} = \frac{1}{\kappa}\ln\frac{yu_\tau}{\nu} + C \ \Rightarrow \ \frac{dU}{dy} = \frac{u_\tau}{\kappa y}$$

The RANS equation (23) in x-direction assuming $\partial/\partial t = \partial/\partial x = V = 0$ and constant P is

$$0 = \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial U}{\partial y}\right]$$

integrating once give

$$(\nu + \nu_T)\frac{\partial U}{\partial y} = u_\tau^2$$

and by applying the log law and neglecting ν

$$C_\mu \frac{K^2}{\varepsilon}\frac{u_\tau}{\kappa y} = u_\tau^2 \qquad (33)$$

$\mathcal{P} \approx \varepsilon$ is assumed in the log law, which gives

$$\varepsilon = \mathcal{P} = \nu_T\left(\frac{\partial U}{\partial y}\right)^2 = C_\mu \frac{K^2}{\varepsilon}\left(\frac{u_\tau}{\kappa y}\right)^2 \qquad (34)$$

Combining (33) and (34) gives the log-law relations

$$K = \frac{u_\tau^2}{\sqrt{C_\mu}} \text{ and } \varepsilon = \frac{u_\tau^3}{\kappa y} \qquad (35)$$

$\mathcal{D} = 0$ since K is constant, and the equation for K is fulfilled by $\mathcal{P} = \varepsilon$.

The equation for ε reads

$$0 = C_{\varepsilon 1}\frac{\varepsilon}{K}\mathcal{P} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} + \frac{d}{dy}\left(\frac{\nu_T}{\sigma_\varepsilon}\frac{d\varepsilon}{dy}\right)$$

and by pluggin in $\mathcal{P} = \varepsilon$ and using (35) gives

$$\sigma_\varepsilon = \frac{\kappa^2}{(C_{\varepsilon 2} - C_{\varepsilon 1})\sqrt{C_\mu}}$$

Using the “standard” values and $\kappa = 0.41$ gives $\sigma_\varepsilon = 1.2$. The standard value is $\sigma_\varepsilon = 1.3$.

Summary of the standard eddy-viscosity $K - \varepsilon$ model

$$\frac{DK}{Dt} = \mathcal{P} - \varepsilon + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_k}\right]$$

$$\frac{D\varepsilon}{Dt} = (C_{\varepsilon 1}\mathcal{P} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{K} + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_\varepsilon}\right)\frac{\partial \varepsilon}{\partial x_k}\right] \qquad (36)$$

$$\mathcal{P} = 2\nu_T S_{ij}S_{ji} \qquad \nu_T = C_\mu\frac{K^2}{\varepsilon}$$

Model coefficients (standard values):

$$C_\mu = 0.09,\ C_{\varepsilon 1} = 1.44,\ C_{\varepsilon 2} = 1.92,\ \sigma_K = 1.0,\ \sigma_\varepsilon = 1.3$$

The eddy-viscosity $K - \omega$ model

Kolmogorov (1942), Wilcox (80:s and 90:s)

ω is intepreted as the inverse time scale of the large eddies, and the $K - \omega$ model reads

$$\frac{DK}{Dt} = 2\nu_T S_{ij}S_{ji} - C_\mu K\omega + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_K}\right)\frac{\partial K}{\partial x_k}\right]$$

$$\frac{D\omega}{Dt} = 2\alpha S_{ij}S_{ji} - \beta\omega^2 + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_T}{\sigma_\omega}\right)\frac{\partial \omega}{\partial x_k}\right] \qquad (37)$$

$$\nu_T = \frac{K}{\omega}$$

The model coefficients proposed by Wilcox (1988) are

$$C_\mu = 0.09, \ \alpha = 5/9 \approx 0.56, \ \beta = 0.075, \ \sigma_K = 2.0, \ \sigma_\omega = 2.0$$

The equation for ω may also be derived from the ε equation using the variable transformation $\omega = \varepsilon / C_\mu K$

$$\frac{D\omega}{Dt} = \frac{D}{Dt}\left(\frac{\varepsilon}{C_\mu K}\right) = \frac{1}{C_\mu}\left(\frac{1}{K}\frac{D\varepsilon}{Dt} - \frac{\varepsilon}{K^2}\frac{DK}{Dt}\right) = \ldots \tag{38}$$

This transformation relates the α and β coefficients to $C_{\varepsilon 1}$ and $C_{\varepsilon 2}$ as

$$C_{\varepsilon 1} = 1 + \alpha = 1.56 \text{ and } C_{\varepsilon 2} = 1 + \frac{\beta}{C_\mu} = 1.83$$

The turbulent diffusion term from the transformed $K-\varepsilon$ model will, however, contain additional terms

$$\mathcal{D}_\varepsilon = \frac{\partial}{\partial x_k}\left[\frac{\nu_T}{\sigma_\varepsilon}\frac{\partial \omega}{\partial x_k}\right] + \frac{\omega}{K}\frac{\partial}{\partial x_k}\left[\nu_T\left(\frac{1}{\sigma_\varepsilon} - \frac{1}{\sigma_K}\right)\frac{\partial K}{\partial x_k}\right] + \frac{2\nu_T}{K\sigma_\varepsilon}\frac{\partial K}{\partial x_k}\frac{\partial \omega}{\partial x_k} \tag{39}$$

which is the major difference between the $K-\varepsilon$ and $K-\omega$ models. Also the coefficients are differently calibrated. Notably, is $\sigma_K = 2.0$ compared with $\sigma_K = 1.0$ in the $K-\varepsilon$ model.

One serious problem with the $K-\omega$ model is that the model is (unphysically) sensitive for the free stream conditions on K and ω. This can be avoided by carefully calibration of the Schmidt numbers and by introducing a "cross diffusion term" (the last term in eq. 39). Such modifications have been proposed by Kok (1999) and Menter (1993).

Boundary conditions

The $K-\varepsilon$ model cannot be applied all the way down to the wall. Since $K \rightarrow 0$ and ε is finite at the wall, the ε equation becomes singular at the wall because of the ε/K term. There are two different solutions of this

- Log-law boundary conditions: The boundary condition is set in the log layer away from the wall according to the log-law relations. The problem is that the log-law is only strictly valid in equilibrium boundary layers and breaks down close to separation and reattachment points. However, the approach still works reasonable well also in separated flows.
- Near-wall corrections (or low-Reynolds number corrections): The $K-\varepsilon$ model equations are modified with "wall damping functions" based on $y^+ \equiv y u_\tau / \nu$, $y^* \equiv y\sqrt{K}/\nu$ or $Re_T \equiv K^2/\nu\varepsilon$ active in the inner part up to

$y^+ \sim 50$. The requirement on the resolution of the computational grid close to the wall is high. $\Delta y^+ \approx 1$ for the first grid points at the wall. A lot of variant are availabe in the litterature.

The $K-\omega$ model can be computed down to the wall and gives reasonable mean velocity profiles without any modifications. The turbulence properties in the viscous sub-layer is, however, not correctly captured. The $K-\omega$ model can also be used together with wall function boundary conditions.

Realizability considerations for two-equation eddy-viscosity models

A physical requirement on the Reynolds stresses is that the normal components (u^2, v^2 and w^2) must be positive in all possible coordinate systems. This means that the Reynolds stress tensor must be positive definite (all eigenvalues must be positive). Translated to the Reynolds stress anisotropy, this means that

$$-\frac{2}{3} \le a_{\alpha\alpha} \le \frac{4}{3}$$

(no summation over Greek indices) and

$$|a_{\alpha\beta}| \le 1 \ (\alpha \ne \beta)$$

The linear eddy-viscosity relation gives

$$a_{ij} = -2C_\mu \frac{K}{\varepsilon} S_{ij}$$

and it is easily understood that the anisotropy can reach large values that are physically unrealizable if S_{ij} is large. In e.g. shear flows the a_{12} component becomes

$$a_{12} = -2C_\mu \sigma, \ \sigma = \frac{1}{2}\frac{K}{\varepsilon}\frac{dU}{dy}$$

which will give unrealizable a_{12} for $\sigma > 1/(2C_\mu) \approx 5.6$.

The production term in the eddy-viscosity assumption is $\mathcal{P} = 2\nu_T S_{ij} S_{ji}$, which is proportional to the strain rate squared, while the exact production is $\mathcal{P} = -K a_{ij} S_{ji}$, linear in S_{ij}. The eddy-viscosity model, thus, gives excessive production in flows with strong strain rate.

In the Menter $K-\omega$ SST model (Menter 1993), the C_μ coefficient is limited for high shear rates (or for $\mathcal{P}/\varepsilon > 1$) in order to limit a_{12} to 0.3 according to the Bradshaw assumption, preserving realizable values for a_{12}. Moreover, it has been observed that in adverse pressure gradient where $\mathcal{P}/\varepsilon > 1$ the limitation of the a_{12} will be active, which will significantly improve the prediction of adverse pressure gradient flows including flow separation.

Differential Reynolds stress models (DRSM)

The exact equation for the Reynolds stress tensor is derived in (9), and reads

$$\frac{D\overline{u_i u_j}}{Dt} = \mathcal{P}_{ij} - \varepsilon_{ij} + \Pi_{ij} + \mathcal{D}_{ij}$$
$$\frac{D\varepsilon}{Dt} = (C_{\varepsilon 1}\mathcal{P} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{K} + \mathcal{D}_\varepsilon \tag{40}$$

where the production term is exact and needs no further modelling

$$\mathcal{P}_{ij} = -\overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} = \varepsilon\left(\frac{4}{3}S_{ij} + a_{ik}S_{kj} + S_{ik}a_{kj} - a_{ik}\Omega_{kj} + \Omega_{ik}a_{kj}\right)$$

The production term is dependent on the strain- and rotation rate tensors (S_{ij} and Ω_{ij}). The dependency on Ω_{ij} clearly improves the prediction of the influence of rotation. Moreover, the production is linearly dependent on the strain rate, and, thus the excessive production seen in eddy-viscosity models are here avoided.

Modelling of the different terms

- Rotta (1951)
- LRR: Launder, Reece and Rodi (1975)
- SSG: Sarkar, Speziale and Gatski (1991)

• The dissipation rate tensor, ε_{ij}

Introduce the dissipation rate anisotropy

$$e_{ij} \equiv \frac{\varepsilon_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij} \tag{41}$$

The dissipation rate $\varepsilon \equiv \varepsilon_{ii}/2$ and is derived by a transport equation for ε, very similar to the ε equation in the $K-\varepsilon$ model. Alternativelly, an ω equation, similar to that in the $K-\omega$ model, can be used.

The simplest model is that the dissipation rate is isotropic, or $e_{ij} = 0$.

• The pressure strain rate term, Π_{ij}

The exact form of Π_{ij} is given by

$$\Pi_{ij} = \frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$$

A poisson equation for the pressure fluctuation is derived by taking the divergence of the equation for the velocity fluctuations. This equation may be formally solved giving the principal form of the model for Π_{ij}. This gives that the pressure strain term is divided into a slow and a rapid part, $\Pi_{ij}^{(s)}$ and $\Pi_{ij}^{(r)}$, and modelled separatley. In the LRR model, the slow part as proposed by Rotta (1951) was modelled as

$$\Pi_{ij}^{(s)} = -c_1 \varepsilon a_{ij}$$

where the Rotta constant $c_1 = 1.5 - 1.8$. The rapid part proposed by LRR

$$\Pi_{ij}^{(r)} = \frac{4}{5}\varepsilon S_{ij} + \frac{9c_2 + 6}{11}\varepsilon\left(a_{ik}S_{kj} + S_{ik}a_{kj} - \frac{2}{3}a_{kl}S_{lk}\delta_{ij}\right) + \frac{-7c_2 + 10}{11}\varepsilon(a_{ik}\Omega_{kj} + \Omega_{ik}a_{kj})$$

with the constant $c_2 = 0.4 - 0.6$.

In later models, the connection to the exact Poisson equation is weaker, and the pressure strain term is lumped together with the dissipation rate anisotropy as

$$\frac{\Pi_{ij}}{\varepsilon} - e_{ij} = f_{ij}\left(a_{ij}, \frac{K}{\varepsilon}S_{ij}, \frac{K}{\varepsilon}\Omega_{ij}\right)$$

- Turbulent diffusion terms, $\mathcal{D}_{ij}$ and $\mathcal{D}_{\varepsilon}$:

In thin shear layers the vertical fluctuations $\overline{v^2}$ are most responsible for the turbulent mixing. A model that better responds to this, compared to the gradient diffusion, is the generalized gradient diffusion model (GGD) proposed by Daly & Harlow (1970)

$$\mathcal{D}_{ij} = \frac{\partial}{\partial x_k}\left[\left(\nu\delta_{kl} - c_s\frac{K}{\varepsilon}\overline{u_k u_l}\right)\frac{\partial \overline{u_k u_l}}{\partial x_l}\right]$$

$$\mathcal{D}_{\varepsilon} = \frac{\partial}{\partial x_k}\left[\left(\nu\delta_{kl} - c_\varepsilon\frac{K}{\varepsilon}\overline{u_k u_l}\right)\frac{\partial \varepsilon}{\partial x_l}\right]$$

In thin shear layers where $\partial/\partial y$ is dominating these will reduce to

$$\mathcal{D}_{ij} = \frac{\partial}{\partial y}\left[\left(\nu - c_s\frac{K}{\varepsilon}\overline{v^2}\right)\frac{\partial \overline{u_k u_l}}{\partial y}\right]$$

$$\mathcal{D}_{\varepsilon} = \frac{\partial}{\partial y}\left[\left(\nu - c_\varepsilon\frac{K}{\varepsilon}\overline{v^2}\right)\frac{\partial \varepsilon}{\partial y}\right]$$

Standar values are $c_s = 0.25$ and $c_\varepsilon = 0.15$.

Explicit algebraic Reynolds stress models (EARSM)

The starting point is the DRSM

$$\frac{D\overline{u_iu_j}}{Dt} - \mathcal{D}_{ij} = \mathcal{P}_{ij} - \varepsilon_{ij} + \Pi_{ij}$$

$$\frac{D\varepsilon}{Dt} - \mathcal{D}_{\varepsilon} = (C_{\varepsilon 1}\mathcal{P} - C_{\varepsilon 2}\varepsilon)\frac{\varepsilon}{K}$$

The equation for $\overline{u_iu_j}$ is rewritten in terms of the anisotropy and K

$$\frac{Da_{ij}}{Dt} = \frac{D}{Dt}\left(\frac{\overline{u_iu_j}}{K} - \frac{2}{3}\delta_{ij}\right) = \frac{1}{K}\frac{D\overline{u_iu_j}}{Dt} - \frac{\overline{u_iu_j}}{K^2}\frac{DK}{Dt} = \dots$$

$$\frac{DK}{Dt} = \frac{1}{2}\frac{D\overline{u_iu_i}}{Dt} = \frac{1}{2}(\mathcal{P}_{ii} - \varepsilon_{ii} + \mathcal{D}_{ii})$$

which gives

$$\frac{K}{\varepsilon}\left(\frac{Da_{ij}}{Dt} - \mathcal{D}_{ij}^{(a)}\right) = -\left(a_{ij} + \frac{2}{3}\delta_{ij}\right)\left(\frac{\mathcal{P}}{\varepsilon} - 1\right) + \frac{\mathcal{P}_{ij} - \varepsilon_{ij} + \Pi_{ij}}{\varepsilon} \qquad (42)$$

The model for Π_{ij} that can be considered in EARSMs must be “quasi”-linear in a_{ij}, that is tensorially linear in a_{ij} (e.g. the LRR model)

$$\frac{\Pi_{ij}}{\varepsilon} - e_{ij} = -\frac{1}{2}\left(C_1^0 + C_1^1\frac{\mathcal{P}}{\varepsilon}\right)a_{ij} + C_2 S_{ij} + \frac{C_3}{2}\left(a_{ik}S_{kj} + S_{ik}a_{kj} - \frac{2}{3}a_{kl}S_{kl}\delta_{ij}\right) \qquad (43)$$
$$- \frac{C_4}{2}(a_{ik}\Omega_{kj} + \Omega_{ik}a_{kj})$$

In this equation, and in the following concerning EARSM, the S_{ij} and Ω_{ij} are normalized by the turbulent time scale K/ε

$$S_{ij} \equiv \frac{1}{2}\frac{K}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \; \Omega_{ij} \equiv \frac{1}{2}\frac{K}{\varepsilon}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$$

The resulting equation for a_{ij}

$$\frac{1}{A_0}\frac{K}{\varepsilon}\left(\frac{Da_{ij}}{Dt} - \mathcal{D}_{ij}^{(a)}\right) = \left(A_3 + A_4\frac{\mathcal{P}}{\varepsilon}\right)a_{ij} + A_1 S_{ij} - (a_{ik}\Omega_{kj} + \Omega_{ik}a_{kj}) \qquad (44)$$
$$+ A_2\left(a_{ik}S_{kj} + S_{ik}a_{kj} - \frac{2}{3}a_{kl}S_{lk}\delta_{ij}\right)$$

and the A coefficients are directly related to the C coefficients.

Weak equilibrium condition

So far, we have only rewritten the equation for $\overline{u_i u_j}$ as an equation for a_{ij} and an equation for K. The major assumption in obtaining an algebraic relation for a_{ij} is the so called weak equilibrium assumption. That is that the anisotropy is assumed to be constant in time and space. Thus, the l.h.s. in (44) vanishes, resulting in

$$N\boldsymbol{a} = -A_1\boldsymbol{S} + (\boldsymbol{a}\Omega - \Omega\boldsymbol{a}) - A_2\left(\boldsymbol{aS} + \boldsymbol{Sa} - \frac{2}{3}\{\boldsymbol{aS}\}\boldsymbol{I}\right) \qquad (45)$$

where

$$N = A_3 + A_4\frac{\mathcal{P}}{\varepsilon} \qquad (46)$$

Here, the bold matrix notation is used where e.g. $\boldsymbol{a}\Omega \equiv a_{ik}\Omega_{kj}$, $\{\boldsymbol{aS}\}$ denotes the trace of $\boldsymbol{aS}$ ($\{\boldsymbol{aS}\} \equiv a_{kl}S_{lk}$) and $\boldsymbol{I} \equiv \delta_{ij}$ is the identity matrix.

The weak equilibrium assumption is:

- exact in steady homogeneous flows (homogeneous shear flows)
- good approximation in fully developed shear flows
- good approximation even in flows with (slowly) varying a_{ij} with high S_{ij} and Ω_{ij} since (44) is then dominating by the source terms on the r.h.s.
- bad approximation in flows with low S_{ij} and Ω_{ij} since (44) is then dominating by the transport terms on the l.h.s.

Formal solution of (44) => EARSM

The algebraic relation (44) (ARSM) is implicit in a_{ij} and also non-linear in a_{ij} since the ratio $\mathcal{P}/\varepsilon \equiv \{\boldsymbol{aS}\}$.

The first step is to solve the tensor equation by considering the $\mathcal{P}/\varepsilon$ ratio (or N) as unknown. The non-linearity will be consider later.

The tensor relation is, thus, that $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{S}, \Omega)$. The most general expression consists of ten tensorially independent terms

$$\boldsymbol{a} = \sum_{i=1}^{10} \beta_i \boldsymbol{T}^{(i)} \qquad (47)$$

where

$$\boldsymbol{T}^{(1)} = \boldsymbol{S} \qquad \boldsymbol{T}^{(2)} = \boldsymbol{S}^2 - \frac{1}{3}II_S\boldsymbol{I}$$

$$\boldsymbol{T}^{(3)} = \Omega^2 - \frac{1}{3}II_\Omega\boldsymbol{I} \qquad \boldsymbol{T}^{(4)} = \boldsymbol{S}\Omega - \Omega\boldsymbol{S}$$

$$\boldsymbol{T}^{(5)} = \boldsymbol{S}^2\Omega - \Omega\boldsymbol{S}^2 \qquad \boldsymbol{T}^{(6)} = \boldsymbol{S}\Omega^2 + \Omega^2\boldsymbol{S} - \frac{2}{3}IV\boldsymbol{I} \tag{48}$$

$$\boldsymbol{T}^{(7)} = \boldsymbol{S}^2\Omega^2 + \Omega^2\boldsymbol{S}^2 - \frac{2}{3}V\boldsymbol{I} \qquad \boldsymbol{T}^{(8)} = \boldsymbol{S}\Omega\boldsymbol{S}^2 - \boldsymbol{S}^2\Omega\boldsymbol{S}$$

$$\boldsymbol{T}^{(9)} = \Omega\boldsymbol{S}\Omega^2 - \Omega^2\boldsymbol{S}\Omega \qquad \boldsymbol{T}^{(10)} = \Omega\boldsymbol{S}^2\Omega^2 - \Omega^2\boldsymbol{S}^2\Omega$$

The β_i coefficients may be functions of the five independent invariants

$$II_S = \{\boldsymbol{S}^2\},\ II_\Omega = \{\Omega^2\},\ III = \{\boldsymbol{S}^3\},\ IV = \{\boldsymbol{S}\Omega^2\},\ V = \{\boldsymbol{S}^2\Omega^2\} \tag{49}$$

and are determined by inserting the ansatz (47) in the ARSM equation (45) and solving the resulting equation system for the β_i coefficients.

The Wallin & Johansson model in 2D flows

In the model proposed by Wallin & Johansson (2000), the A_2 coefficient is choosen to be 0 in (45), which corresponds to the specific choice of $C_2 = 2$ in the pressure strain rate model (43). With $A_2 = 0$ only five of the ten terms in (48) will remain, and the algebra is, thus, significantly reduced.

The algebra is rather complex for 3D mean flows, and, thus, the method will here be described in 2D mean flows, where the number of independent tensor groups in general are three, but with $A_2 = 0$ they reduces to two: $\boldsymbol{T}^{(1)}$ and $\boldsymbol{T}^{(4)}$. Also only two independent invariants remain, II_S and II_Ω. The Wallin & Johansson model is also build on the LRR model, slightly recalibrated, with $c_1 = 1.8$ and $c_2 = 5/9 \approx 0.56$, which gives the following model coefficients

$$A_0 = -\frac{4}{9},\ A_1 = \frac{6}{5},\ A_2 = 0,\ A_3 = \frac{9}{4}(c_1 - 1),\ A_4 = \frac{9}{4}$$

The implicit algebraic relation (45) becomes

$$N\boldsymbol{a} = -\frac{6}{5}\boldsymbol{S} + (\boldsymbol{a}\Omega - \Omega\boldsymbol{a}) \tag{50}$$

and with the ansatz (47)

$$\boldsymbol{a} = \beta_1\boldsymbol{S} + \beta_4(\boldsymbol{S}\Omega - \Omega\boldsymbol{S}) \tag{51}$$

it becomes

$$N(\beta_1\boldsymbol{S} + \beta_4(\boldsymbol{S}\Omega - \Omega\boldsymbol{S})) = \tag{52}$$
$$-\frac{6}{5}\boldsymbol{S} + \beta_1(\boldsymbol{S}\Omega - \Omega\boldsymbol{S}) + \beta_4(\boldsymbol{S}\Omega^2 - 2\Omega\boldsymbol{S}\Omega + \Omega^2\boldsymbol{S})$$

In 2D mean flows, the term

$$\boldsymbol{S}\Omega^2 - 2\Omega\boldsymbol{S}\Omega + \Omega^2\boldsymbol{S} = 2II_\Omega\boldsymbol{S}$$

and (52) becomes

$$N(\beta_1\boldsymbol{S} + \beta_4(\boldsymbol{S}\Omega - \Omega\boldsymbol{S})) = -\frac{6}{5}\boldsymbol{S} + \beta_1(\boldsymbol{S}\Omega - \Omega\boldsymbol{S}) + 2\beta_4 II_\Omega\boldsymbol{S} \tag{53}$$

sorting equal tensor terms gives

$$\begin{aligned} N\beta_1 &= -\frac{6}{5} + 2\beta_4 II_\Omega \\ N\beta_4 &= \beta_1 \end{aligned} \quad \Rightarrow \quad \beta_1 = N\beta_4 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega} \tag{54}$$

and the solution for a_{ij} is given with (54) in (52) as

$$\boldsymbol{a} = -\frac{6}{5}\frac{(N\boldsymbol{S} + \boldsymbol{S}\Omega - \Omega\boldsymbol{S})}{N^2 - 2II_\Omega} \tag{55}$$

The consistency condition

The production to dissipation ratio, $\mathcal{P}/\varepsilon$, can be written as

$$\frac{\mathcal{P}}{\varepsilon} = \frac{K a_{ij} S'_{ji}}{\varepsilon} = a_{ij} S_{ji} = \{\boldsymbol{a}\boldsymbol{S}\} \tag{56}$$

Using the solution (52) for the anisotropy this gives that

$$\frac{\mathcal{P}}{\varepsilon} = \beta_1\{\boldsymbol{S}^2\} + \beta_4(\{\boldsymbol{S}\Omega\boldsymbol{S}\} - \{\Omega\boldsymbol{S}^2\}) = \beta_1 II_S \tag{57}$$

since $\{\boldsymbol{S}\Omega\boldsymbol{S}\} = \{\Omega\boldsymbol{S}^2\} = 0$. Using the solution for β_1 in (54) and the relation for N in (46)

$$\frac{4}{9}(N - A_3) = -\frac{6}{5}\frac{NII_S}{N^2 - 2II_\Omega} \tag{58}$$

and rearranged

$$N^3 - A_3N^2 - \left(\frac{27}{10}II_S + 2II_\Omega\right)N + 2A_3II_\Omega = 0 \tag{59}$$

Closed solutions can be obtained from cubic equations

$$N = \begin{cases} \dfrac{A_3}{3} + (P_1 + \sqrt{P_2})^{1/3} + (P_1 - \sqrt{P_2})^{1/3} & P_2 \ge 0 \\ \dfrac{A_3}{3} + 2(P_1^2 - P_2)^{1/6}\cos\left(\dfrac{1}{3}\mathrm{acos}\left(\dfrac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right) & P_2 < 0 \end{cases} \tag{60}$$

$$P_1 = \left(\frac{A_3^2}{27} + \frac{9}{20}II_S - \frac{2}{3}II_\Omega\right)A_3 \qquad P_2 = P_1^2 - \left(\frac{A_3^2}{9} + \frac{9}{10}II_S + \frac{2}{3}II_\Omega\right)^3$$

Effective C_μ

The first term in the solution (51) corresponds to the eddy-viscosity assumption where

$$C_\mu^{\mathrm{eff}} = -\frac{\beta_1}{2} = \frac{3}{5}\frac{N}{N^2 - 2II_\Omega} \tag{61}$$

here, the C_μ is not a constant, thus the C_μ^{eff}, and contain rotational effects as well as the possibility to adjust to the local flow conditions.

The boundary layer log region

Since $\mathcal{P} = \varepsilon$, N can be determined directly as

$$N = A_3 + A_4\frac{\mathcal{P}}{\varepsilon} = \frac{9}{4}\left(c_1 - 1 + \frac{\mathcal{P}}{\varepsilon}\right) = \frac{9c_1}{4} = 4.05 \tag{62}$$

In parallell flows

$$II_S = -II_\Omega = \frac{1}{2}\left(\frac{K}{\varepsilon}\frac{dU}{dy}\right)^2 \tag{63}$$

and the $\mathcal{P}/\varepsilon$ ratio is derived to

$$\frac{\mathcal{P}}{\varepsilon} = -\beta_1 II_S = \frac{6}{5}\frac{NII_S}{N^2 + 2II_S} = 1 \Rightarrow II_S = \frac{N^2}{\frac{6N}{5} - 2} \approx 5.74 \Rightarrow \frac{K}{\varepsilon}\frac{dU}{dy} = 3.39$$

also

$$\beta_1 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega} \approx -0.174,\ \beta_4 = -\frac{6}{5}\frac{1}{N^2 - 2II_\Omega} \approx -0.043,\ C_\mu^{\mathrm{eff}} = 0.087.$$

The Reynolds stress anisotropy can now be derived

	a_{12}	a_{11}	a_{22}	a_{33}	σ
DNS	–0.29	0.34	–0.26	–0.08	1.65
WJ	–0.30	0.25	–0.25	0	1.69

, where $\sigma \equiv \frac{1}{2}\frac{K}{\varepsilon}\frac{dU}{dy}$

The a_{12} and a_{22} anisotropies and σ are well predicted, but the a_{11} and a_{33} components are not that well. The reason is that $c_2 = 5/9$ which gives that $a_{33} = 0$ in thin shear flows. The approximation is not that problematic, since the a_{12} and a_{22} components are the only components that enters into the balance equations.

High shear rates in parallell flow (homogeneous shear)

For high shear rates, $\sigma \to \infty$ and the following holds

$$P_1 \to \sigma^2,\ P_2 \to -\sigma^6,\ N \to \sigma,\ \beta_1 \to \sigma^{-1},\ \beta_4 \to \sigma^{-2}$$

The anisotropy goes like

$$\boldsymbol{a} = \beta_1 \boldsymbol{S} + \beta_4(\boldsymbol{S}\Omega - \Omega\boldsymbol{S}) \to \sigma^{-1}\sigma + \sigma^{-2}\sigma^2 = \text{const}$$

and the anisotropy will stay realizable (c.f. eddy viscosity models)

The production

$$\mathcal{P} = \beta_1 II_S \to \sigma^{-1}\sigma^2 = \sigma$$

and has the correct development

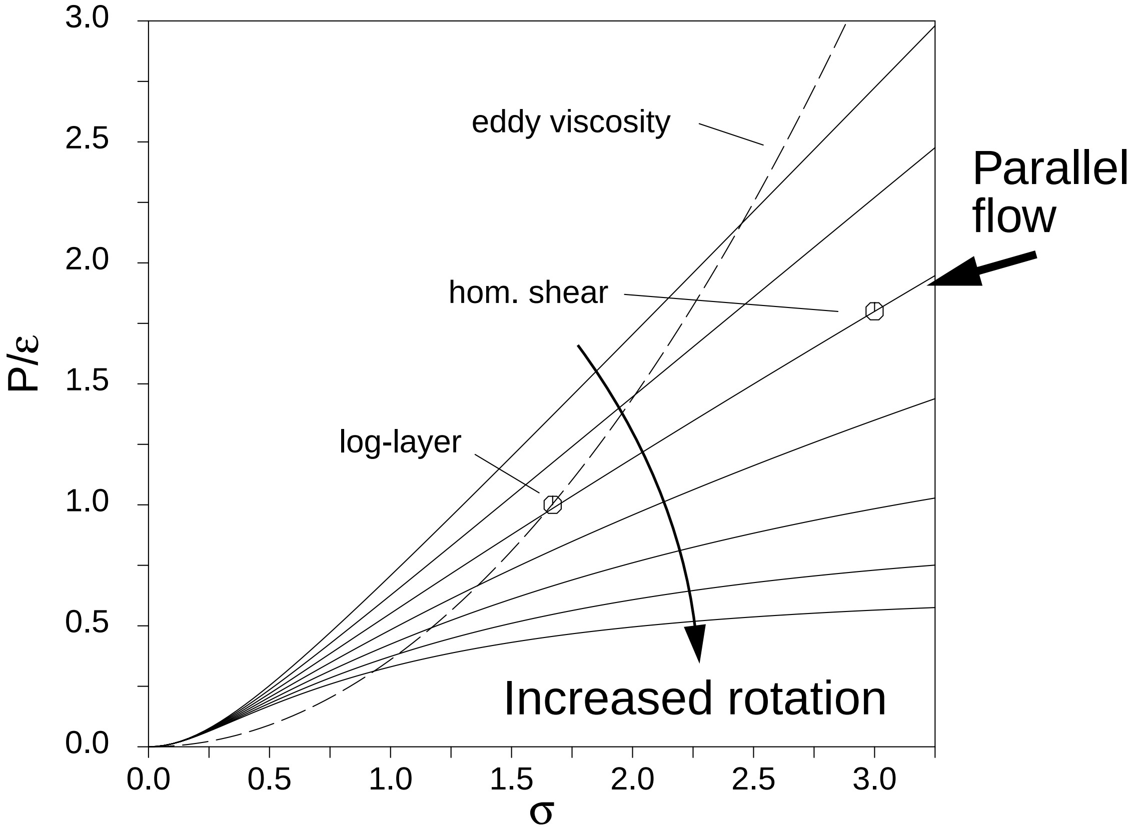
Rotational flows (rotating channel flow)

The rotational effects enter mainly into β_1 through the dependency on II_Ω

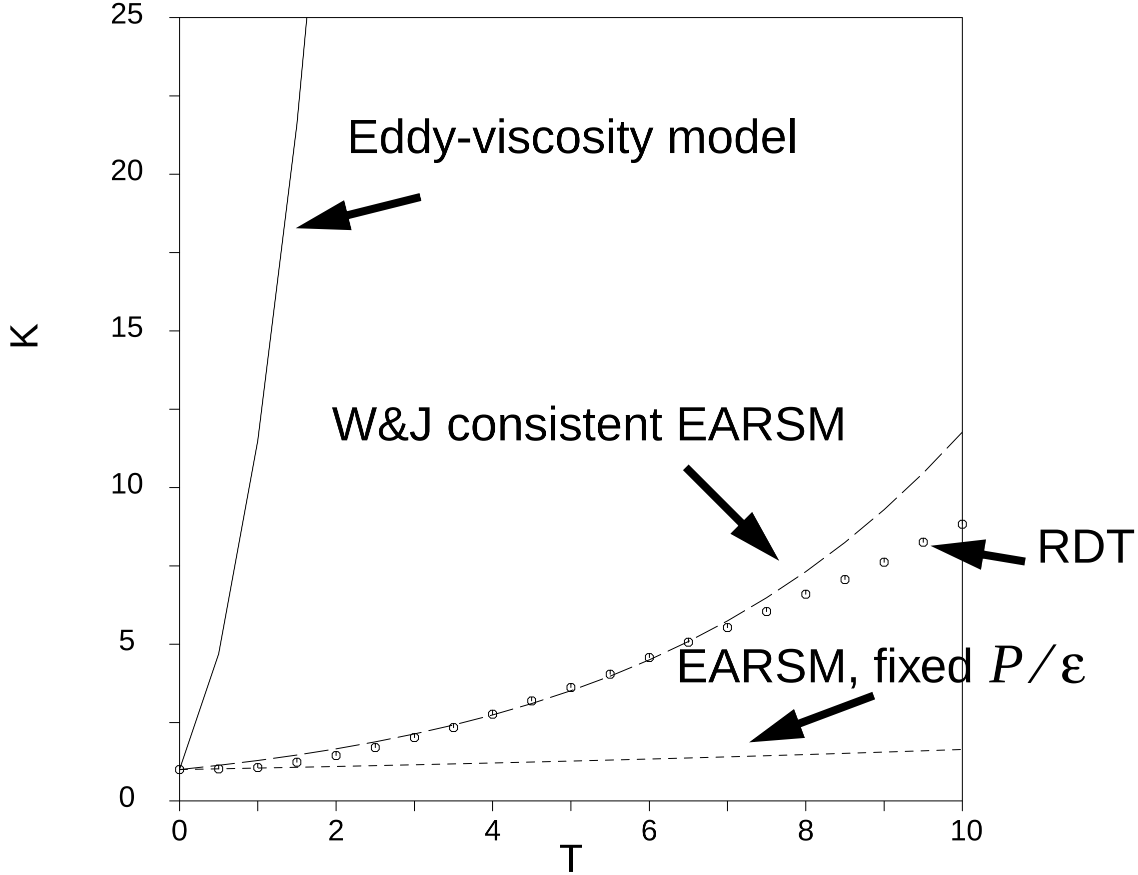
$$\beta_1 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega}$$

In rotating channel flows the system rotation is added to the Ω_{ij} tensor, used for II_Ω. On the unstable side, the system rotation will decrease II_Ω in magnitude (remember $II_\Omega < 0$) and β_1 will increase. The same holds in curved flows, where a concave (unstable) curvature will decrease II_Ω in magnitude and β_1 will increase.

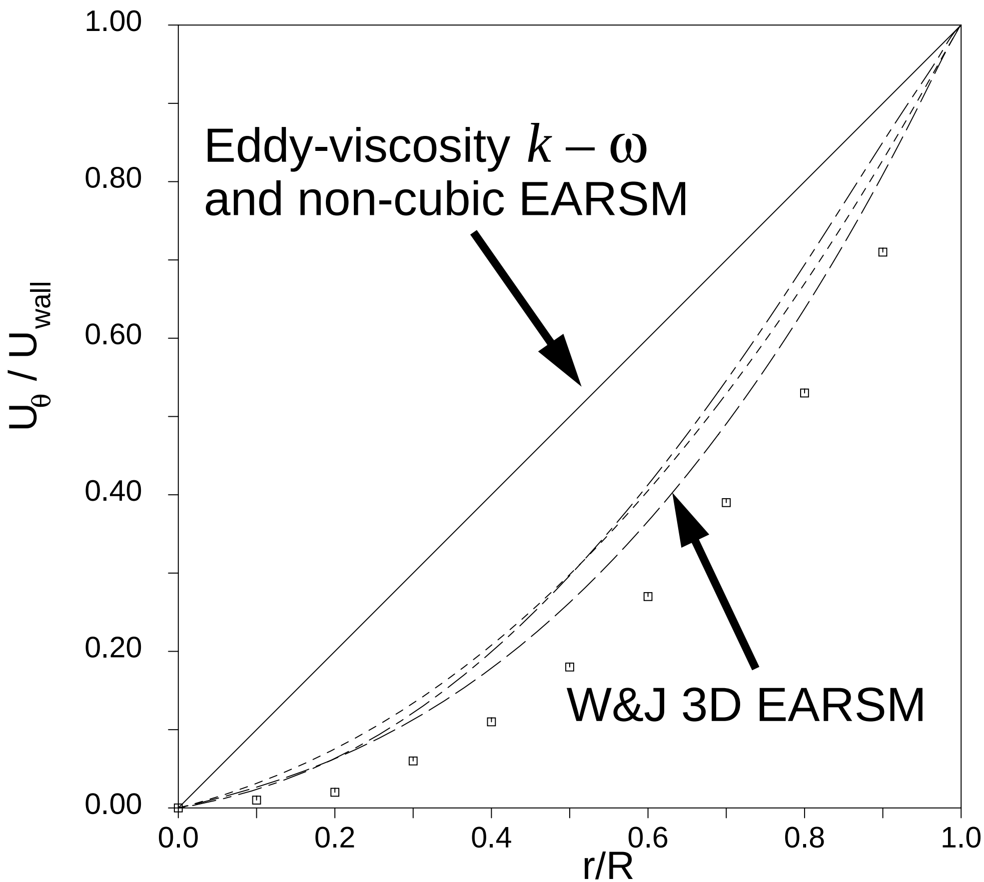
Production to dissipation ratio



Rapidly sheared homogeneous flow



(i) 3D mean flows: Rotating pipe, swirl velocity



(ii) normal stresses

Homogeneous shear:

Model	a_{12}	a_{11}	a_{22}	a_{33}	P/ε
Tavoularis & Corrsin 1981	-0.30	0.40	-0.28	-0.12	1.8
Recalibrated LRR (J&W)	-0.30	0.31	-0.31	0	1.8
Linearized SSG	-0.32	0.41	-0.30	-0.11	1.9

Boundary layers (log layer):

Model	a_{12}	a_{11}	a_{22}	a_{33}	$\frac{SK}{2\varepsilon}$
Moser, Kim & Mansour 1998	-0.29	0.34	-0.26	-0.08	1.65
Recalibrated LRR (J&W)	-0.30	0.25	-0.25	0	1.69
Linearized SSG	-0.32	0.36	-0.26	-0.10	1.59

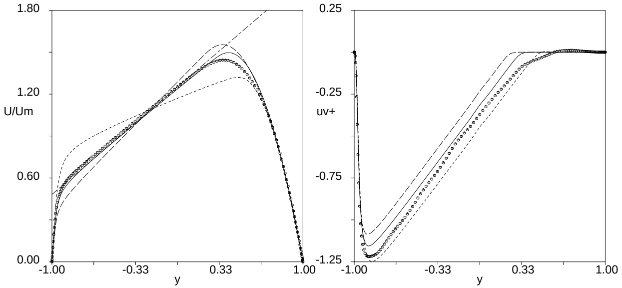


Figure 1: Computed rotating channel flow for $Ro = 0.77$ compared to DNS of Alvelius & Johansson (2000). Curvature corrected original (– – –) and recalibrated (——) WJ EARSM compared to the non-corrected EARSM (- - - - - -). $U' = 2\omega_z^{(r)}$ (– - – -) is also shown. (figure taken from Wallin & Johansson, 2002)

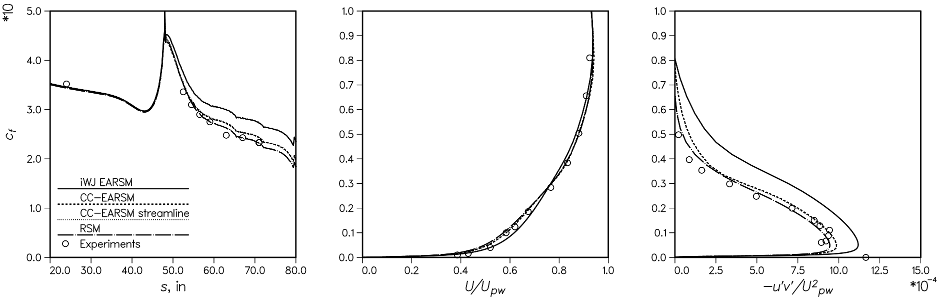


Figure 2: The skin-friction coefficient along the convex wall, the velocity profile and turbulent shear stress at $s = 71$ in. Experiment by So & Mellor (1973). The stresses are transformed into the local wall-tangential and -normal coordinate system, and U_{pw} is the theoretical potential velocity on the wall.