بسمه تعالی

مبحث ممان اینرسی

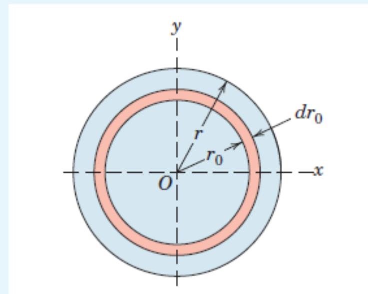
The design of these structural members requires calculation of their cross-sectional moment of inertia. In this chapter we will discuss how this is done.

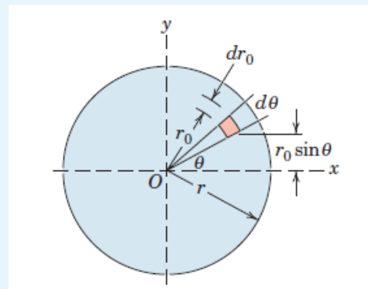
Ref. Engineering mechanics, Statics, Thirteenth edition, R. C. Hibbler, 2013.

مثالهای تکمیلی

SAMPLE PROBLEM A/3

Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.





Solution. A differential element of area in the form of a circular ring may be used for the calculation of the moment of inertia about the polar z-axis through O since all elements of the ring are equidistant from O. The elemental area is $dA = 2\pi r_0\, dr_0$, and thus, ①

$$\left[I_z = \int r^2\, dA\right] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \tfrac{1}{2}Ar^2 \qquad \textit{Ans.}$$

The polar radius of gyration is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \tfrac{1}{2}I_z = \frac{\pi r^4}{4} = \tfrac{1}{4}Ar^2 \qquad \textit{Ans.}$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2} \qquad \textit{Ans.}$$

The foregoing determination of I_x is the simplest possible. The result may also be obtained by direct integration, using the element of area $dA = r_0\, dr_0\, d\theta$ shown in the lower figure. By definition

$$\left[I_x = \int y^2\, dA\right] \qquad I_x = \int_0^{2\pi}\int_0^r (r_0 \sin\theta)^2 r_0\, dr_0\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta$$

② $$= \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4} \qquad \textit{Ans.}$$

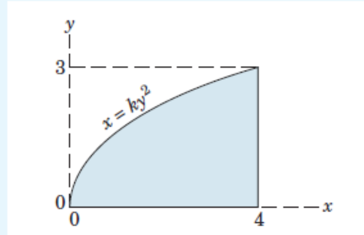
Helpful Hints

① Polar coordinates are certainly indicated here. Also, as before, we choose the simplest and lowest-order element possible, which is the differential ring. It should be evident immediately from the definition that the polar moment of inertia of the ring is its area $2\pi r_0\, dr_0$ times $r_0^{\,2}$.

② This integration is straightforward, but the use of Eq. A/3 along with the result for I_z is certainly simpler.

SAMPLE PROBLEM A/4

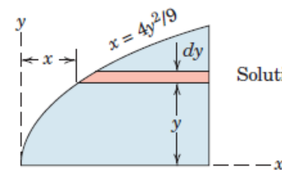
Determine the moment of inertia of the area under the parabola about the x-axis. Solve by using (a) a horizontal strip of area and (b) a vertical strip of area.



Solution. The constant $k = \frac{4}{9}$ is obtained first by substituting $x = 4$ and $y = 3$ into the equation for the parabola.

(a) Horizontal strip. Since all parts of the horizontal strip are the same distance from the x-axis, the moment of inertia of the strip about the x-axis is $y^2\,dA$ where $dA = (4 - x)\,dy = 4(1 - y^2/9)\,dy$. Integrating with respect to y gives us

$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^3 4y^2\left(1 - \frac{y^2}{9}\right)dy = \frac{72}{5} = 14.4\ (\text{units})^4 \qquad \textit{Ans.}$$



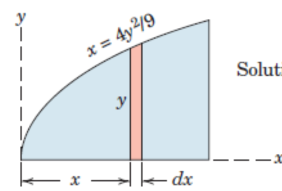
Solution (a)

(b) Vertical strip. Here all parts of the element are at different distances from the x-axis, so we must use the correct expressions for the moment of inertia of the elemental rectangle about its base, which, from Sample Problem A/1, is $bh^3/3$. For the width dx and the height y the expression becomes

$$dI_x = \tfrac{1}{3}(dx)y^3$$

To integrate with respect to x, we must express y in terms of x, which gives $y = 3\sqrt{x}/2$, and the integral becomes

1 $$I_x = \frac{1}{3}\int_0^4 \left(\frac{3\sqrt{x}}{2}\right)^3 dx = \frac{72}{5} = 14.4\ (\text{units})^4 \qquad \textit{Ans.}$$



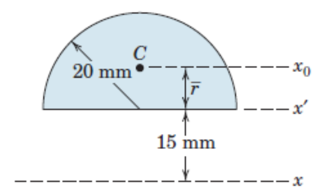
Solution (b)

Helpful Hint

1 There is little preference between Solutions (a) and (b). Solution (b) requires knowing the moment of inertia for a rectangular area about its base.

SAMPLE PROBLEM A/5

Find the moment of inertia about the x-axis of the semicircular area.



Solution. The moment of inertia of the semicircular area about the x'-axis is one-half of that for a complete circle about the same axis. Thus, from the results of Sample Problem A/3,

$$I_{x'} = \frac{1}{2}\frac{\pi r^4}{4} = \frac{20^4\pi}{8} = 2\pi(10^4)\ \text{mm}^4$$

We obtain the moment of inertia $\bar{I}$ about the parallel centroidal axis x_0 next. Transfer is made through the distance $\bar{r} = 4r/(3\pi) = (4)(20)/(3\pi) = 80/(3\pi)$ mm by the parallel-axis theorem. Hence,

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = 2(10^4)\pi - \left(\frac{20^2\pi}{2}\right)\left(\frac{80}{3\pi}\right)^2 = 1.755(10^4)\ \text{mm}^4$$

1 Finally, we transfer from the centroidal x_0-axis to the x-axis. Thus,

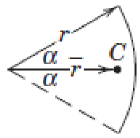
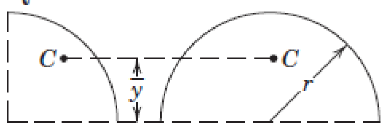
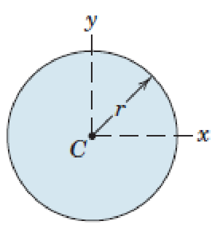
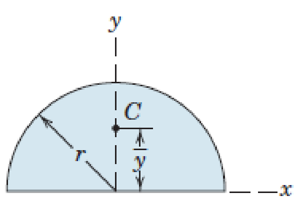
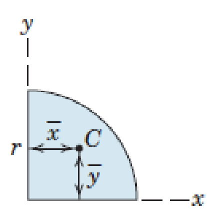
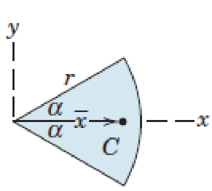
$$[I = \bar{I} + Ad^2] \qquad I_x = 1.755(10^4) + \left(\frac{20^2\pi}{2}\right)\left(15 + \frac{80}{3\pi}\right)^2$$

$$= 1.755(10^4) + 34.7(10^4) = 36.4(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

Helpful Hint

1 This problem illustrates the caution we should observe in using a double transfer of axes since neither the x'- nor the x-axis passes through the centroid C of the area. If the circle were complete with the centroid on the x' axis, only one transfer would be needed.

TABLE D/3 PROPERTIES OF PLANE FIGURES

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r \sin\alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3}\dfrac{r \sin\alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}(\alpha - \dfrac{1}{2}\sin 2\alpha)$ $I_y = \dfrac{r^4}{4}(\alpha + \dfrac{1}{2}\sin 2\alpha)$ $I_z = \dfrac{1}{2} r^4 \alpha$

EXAMPLE 10.4

Determine the moment of inertia of the area shown in Fig. 10–8*a* about the *x* axis.

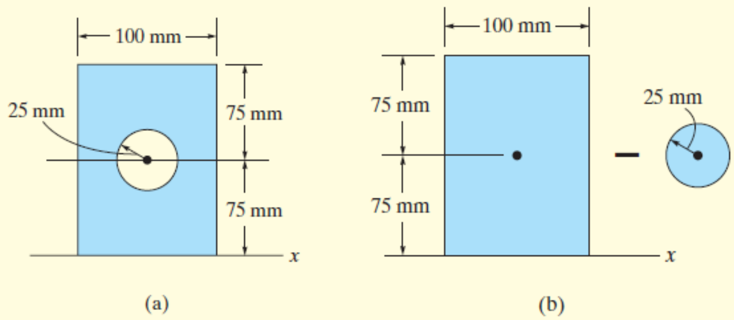


Fig. 10–8

SOLUTION

Composite Parts. The area can be obtained by *subtracting* the circle from the rectangle shown in Fig. 10–8*b*. The centroid of each area is located in the figure.

Parallel-Axis Theorem. The moments of inertia about the x axis are determined using the parallel-axis theorem and the geometric properties formulae for circular and rectangular areas $I_x = \frac{1}{4}\pi r^4$; $I_x = \frac{1}{12}bh^3$, found on the inside back cover.

Circle

$$I_x = \bar{I}_{x'} + Ad_y^2$$
$$= \frac{1}{4}\pi(25)^4 + \pi(25)^2(75)^2 = 11.4(10^6)\text{ mm}^4$$

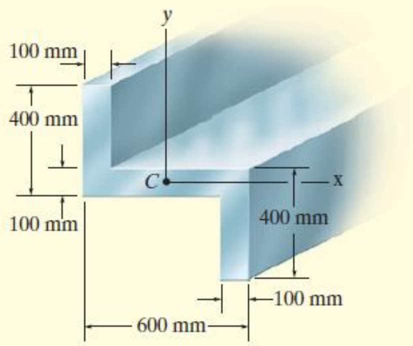
Rectangle

$$I_x = \bar{I}_{x'} + Ad_y^2$$
$$= \frac{1}{12}(100)(150)^3 + (100)(150)(75)^2 = 112.5(10^6)\text{ mm}^4$$

Summation. The moment of inertia for the area is therefore

$$I_x = -11.4(10^6) + 112.5(10^6)$$
$$= 101(10^6)\text{ mm}^4 \qquad \textit{Ans.}$$

EXAMPLE 10.5

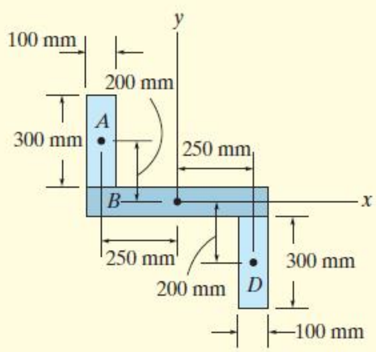


(a)

Determine the moments of inertia for the cross-sectional area of the member shown in Fig. 10–9*a* about the *x* and *y* centroidal axes.

SOLUTION

Composite Parts. The cross section can be subdivided into the three rectangular areas *A*, *B*, and *D* shown in Fig. 10–9*b*. For the calculation, the centroid of each of these rectangles is located in the figure.



(b)

Fig. 10–9

Parallel-Axis Theorem. From the table on the inside back cover, or Example 10.1, the moment of inertia of a rectangle about its centroidal axis is $\bar{I} = \frac{1}{12}bh^3$. Hence, using the parallel-axis theorem for rectangles *A* and *D*, the calculations are as follows:

Rectangles A and D

$$I_x = \bar{I}_{x'} + Ad_y^2 = \frac{1}{12}(100)(300)^3 + (100)(300)(200)^2$$
$$= 1.425(10^9)\text{ mm}^4$$
$$I_y = \bar{I}_{y'} + Ad_x^2 = \frac{1}{12}(300)(100)^3 + (100)(300)(250)^2$$
$$= 1.90(10^9)\text{ mm}^4$$

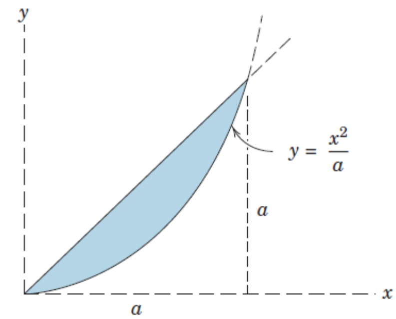
Rectangle B

$$I_x = \frac{1}{12}(600)(100)^3 = 0.05(10^9)\text{ mm}^4$$
$$I_y = \frac{1}{12}(100)(600)^3 = 1.80(10^9)\text{ mm}^4$$

Summation. The moments of inertia for the entire cross section are thus

$$I_x = 2[1.425(10^9)] + 0.05(10^9)$$
$$= 2.90(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$
$$I_y = 2[1.90(10^9)] + 1.80(10^9)$$
$$= 5.60(10^9)\text{ mm}^4 \qquad \textit{Ans.}$$

تمرین ۱- ممان اینرسی سطوح نشان داده شده را حول محور x و y تعیین کنید.



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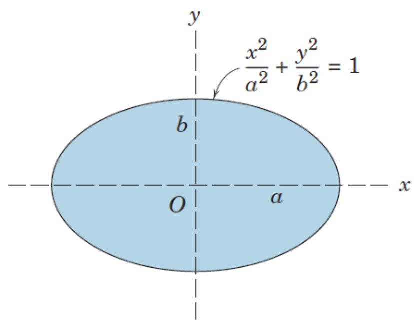
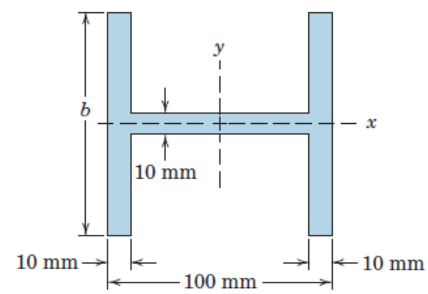


(ب)

تمرین ۲- ممان اینرسی سطح مقطع تیرهای نشان داده شده را حول محور x و y بیابید.



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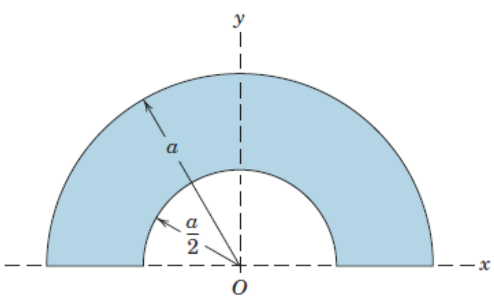
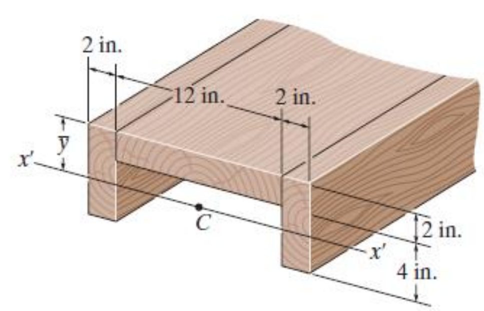


(ب)

تمرین ۳- در شکلهای زیر محور x' از مرکز سطح مقطع نشان داده شده عبور میکند. ممان اینرسی سطح را حول این محور بیابید.



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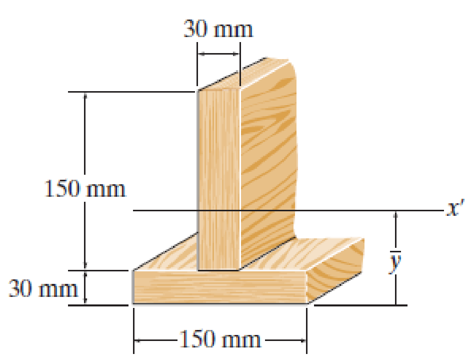
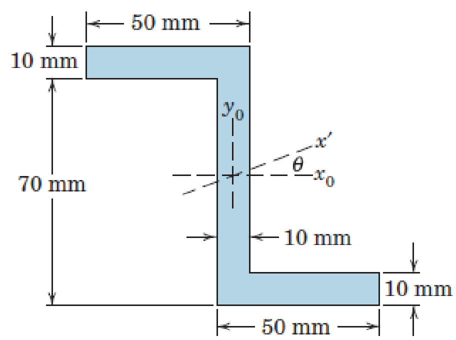


(ب)

تمرین ۴- در مقطع نشان داده شده در شکل، ابتدا گشتاورهای اینرسی سطح حول محورهای گذرنده از مرکز سطح را محاسبه نمایید. سپس به کمک چرخش دستگاه مختصات، مقادیر بیشترین و کمترین گشتاورهای اینرسی سطح و زاویه مربوط به آنرا به دو روش زیر محاسبه نمایید:

الف: به کمک روابط مربوطه

ب: به کمک رسم دایره مور



موفق باشید.