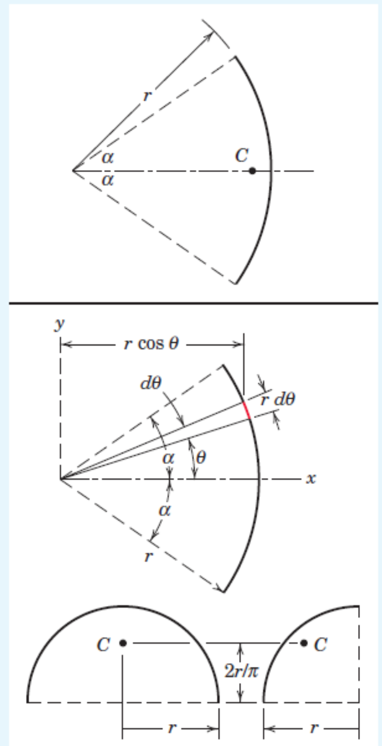
فصل پنجم- بخش اول: مرکز جرم- مرکز هندسی

مثالهای تکمیلی:

SAMPLE PROBLEM 5/1

Centroid of a circular arc. Locate the centroid of a circular arc as shown in the figure.



Solution. Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length $dL = r\,d\theta$ expressed in polar coordinates, and the x-coordinate of the element is $r \cos \theta$. (1)

Applying the first of Eqs. 5/4 and substituting $L = 2\alpha r$ give

$$[L\bar{x} = \int x\,dL] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r \cos \theta)\, r\,d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin \alpha$$

$$\bar{x} = \frac{r \sin \alpha}{\alpha} \qquad \textit{Ans.}$$

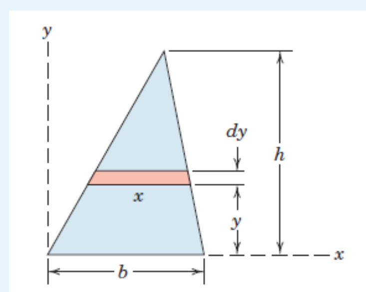
For a semicircular arc $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$. By symmetry we see immediately that this result also applies to the quarter-circular arc when the measurement is made as shown.

Helpful Hint

(1) It should be perfectly evident that polar coordinates are preferable to rectangular coordinates to express the length of a circular arc.

SAMPLE PROBLEM 5/2

Centroid of a triangular area. Determine the distance $\bar{h}$ from the base of a triangle of altitude h to the centroid of its area.



Solution. The x-axis is taken to coincide with the base. A differential strip of area $dA = x\,dy$ is chosen. By similar triangles $x/(h - y) = b/h$. Applying the second of Eqs. 5/5a gives (1)

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\,\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

and

$$\bar{y} = \frac{h}{3} \qquad \textit{Ans.}$$

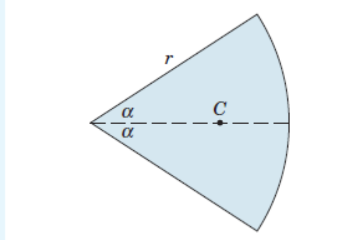
This same result holds with respect to either of the other two sides of the triangle considered a new base with corresponding new altitude. Thus, the centroid lies at the intersection of the medians, since the distance of this point from any side is one-third the altitude of the triangle with that side considered the base.

Helpful Hint

(1) We save one integration here by using the first-order element of area. Recognize that dA must be expressed in terms of the integration variable y; hence, $x = f(y)$ is required.

SAMPLE PROBLEM 5/3

Centroid of the area of a circular sector. Locate the centroid of the area of a circular sector with respect to its vertex.



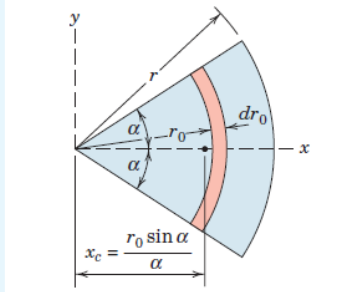
Solution I. The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is ① $dA = 2r_0\alpha\, dr_0$.

The x-coordinate to the centroid of the element from Sample Problem 5/1 is ② $x_c = r_0 \sin\alpha/\alpha$, where r_0 replaces r in the formula. Thus, the first of Eqs. 5/5a gives

$$\left[A\bar{x} = \int x_c\, dA\right] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\bar{x} = \tfrac{2}{3}r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha} \qquad \textit{Ans.}$$



Solution I

Solution II. The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (r/2)(r\, d\theta)$, where higher-order terms are neglected. From Sample Problem 5/2 the centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is $x_c = \frac{2}{3}r\cos\theta$. Applying the first of Eqs. 5/5a gives

$$\left[A\bar{x} = \int x_c\, dA\right] \qquad (r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} (\tfrac{2}{3}r\cos\theta)(\tfrac{1}{2}r^2\, d\theta)$$

$$r^2\alpha\bar{x} = \tfrac{2}{3}r^3 \sin\alpha$$

and as before

$$\bar{x} = \frac{2}{3}\frac{r \sin\alpha}{\alpha} \qquad \textit{Ans.}$$

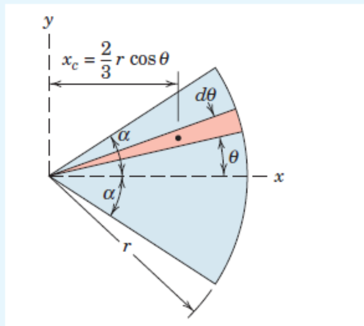
For a semicircular area $2\alpha = \pi$, which gives $\bar{x} = 4r/3\pi$. By symmetry we see immediately that this result also applies to the quarter-circular area where the measurement is made as shown.

It should be noted that, if we had chosen a second-order element $r_0\, dr_0\, d\theta$, one integration with respect to θ would yield the ring with which *Solution I* began. On the other hand, integration with respect to r_0 initially would give the triangular element with which *Solution II* began.

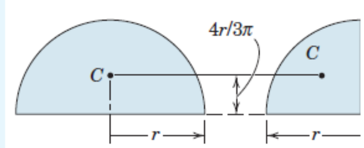
Helpful Hints

① Note carefully that we must distinguish between the variable r_0 and the constant r.

② Be careful not to use r_0 as the centroidal coordinate for the element.

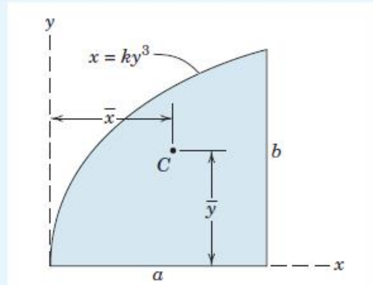


Solution II

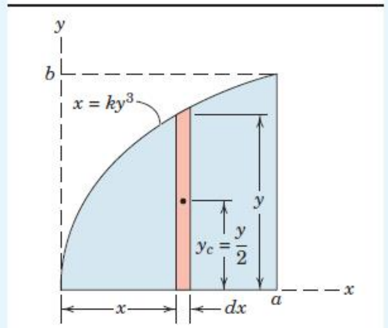


SAMPLE PROBLEM 5/4

Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.



Solution I. A vertical element of area $dA = y\,dx$ is chosen as shown in the figure. The x-coordinate of the centroid is found from the first of Eqs. 5/5a. Thus,



1 $\left[A\bar{x} = \int x_c\,dA\right]$ $$\bar{x}\int_0^a y\,dx = \int_0^a xy\,dx$$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating give

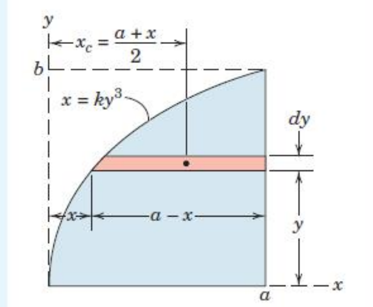
$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \tfrac{4}{7}a \qquad \textit{Ans.}$$

In the solution for $\bar{y}$ from the second of Eqs. 5/5a, the coordinate to the centroid of the rectangular element is $y_c = y/2$, where y is the height of the strip governed by the equation of the curve $x = ky^3$. Thus, the moment principle becomes

$\left[A\bar{y} = \int y_c\,dA\right]$ $$\frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx$$

Substituting $y = b(x/a)^{1/3}$ and integrating give

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \tfrac{2}{5}b \qquad \textit{Ans.}$$



Solution II. The horizontal element of area shown in the lower figure may be employed in place of the vertical element. The x-coordinate to the centroid of the rectangular element is seen to be $x_c = x + \frac{1}{2}(a - x) = (a + x)/2$, which is simply the average of the coordinates a and x of the ends of the strip. Hence,

$\left[A\bar{x} = \int x_c\,dA\right]$ $$\bar{x}\int_0^b (a - x)\,dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\,dy$$

The value of $\bar{y}$ is found from

$\left[A\bar{y} = \int y_c\,dA\right]$ $$\bar{y}\int_0^b (a - x)\,dy = \int_0^b y(a - x)\,dy$$

where $y_c = y$ for the horizontal strip. The evaluation of these integrals will check the previous results for $\bar{x}$ and $\bar{y}$.

Helpful Hint

1 Note that $x_c = x$ for the vertical element.

SAMPLE PROBLEM 5/5

Hemispherical volume. Locate the centroid of the volume of a hemisphere of radius r with respect to its base.

Solution I. With the axes chosen as shown in the figure, $\bar{x} = \bar{z} = 0$ by symmetry. The most convenient element is a circular slice of thickness dy parallel to the x-z plane. Since the hemisphere intersects the y-z plane in the circle $y^2 + z^2 = r^2$, the radius of the circular slice is $z = +\sqrt{r^2 - y^2}$. The volume of the elemental slice becomes

1 $$dV = \pi(r^2 - y^2)\,dy$$

The second of Eqs. 5/6a requires

$$\left[V\bar{y} = \int y_c\,dV\right] \qquad \bar{y}\int_0^r \pi(r^2 - y^2)\,dy = \int_0^r y\pi(r^2 - y^2)\,dy$$

where $y_c = y$. Integrating gives

$$\tfrac{2}{3}\pi r^3\bar{y} = \tfrac{1}{4}\pi r^4 \qquad \bar{y} = \tfrac{3}{8}r \qquad \textit{Ans.}$$

Solution II. Alternatively we may use for our differential element a cylindrical shell of length y, radius z, and thickness dz, as shown in the lower figure. By expanding the radius of the shell from zero to r, we cover the entire volume. By symmetry the centroid of the elemental shell lies at its center, so that $y_c = y/2$. The volume of the element is $dV = (2\pi z\,dz)(y)$. Expressing y in terms of z from the equation of the circle gives $y = +\sqrt{r^2 - z^2}$. Using the value of $\frac{2}{3}\pi r^3$ computed in *Solution I* for the volume of the hemisphere and substituting in the second of Eqs. 5/6a give us

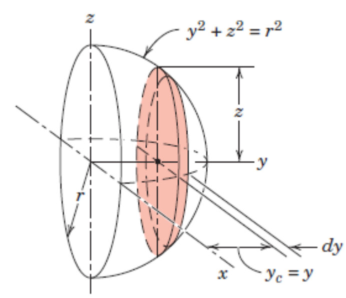
$$\left[V\bar{y} = \int y_c\,dV\right] \qquad (\tfrac{2}{3}\pi r^3)\bar{y} = \int_0^r \frac{\sqrt{r^2 - z^2}}{2}\,(2\pi z\sqrt{r^2 - z^2})\,dz$$

$$= \int_0^r \pi(r^2 z - z^3)\,dz = \frac{\pi r^4}{4}$$

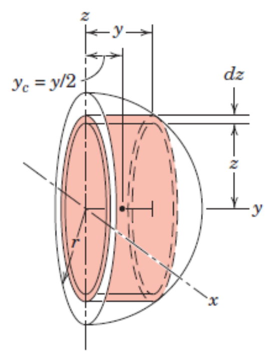
$$\bar{y} = \tfrac{3}{8}r \qquad \textit{Ans.}$$

Solutions I and *II* are of comparable use since each involves an element of simple shape and requires integration with respect to one variable only.

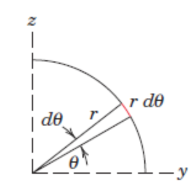
Solution III. As an alternative, we could use the angle θ as our variable with limits of 0 and $\pi/2$. The radius of either element would become $r \sin\theta$, whereas the thickness of the slice in *Solution I* would be $dy = (r\,d\theta)\sin\theta$ and that of the shell in *Solution II* would be $dz = (r\,d\theta)\cos\theta$. The length of the shell would be $y = r\cos\theta$.



Solution I



Solution II



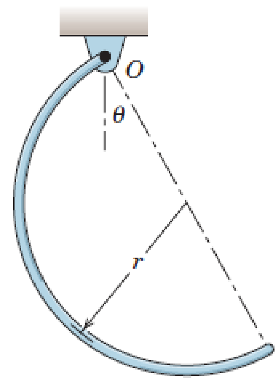
Solution III

Helpful Hint

1 Can you identify the higher-order element of volume which is omitted from the expression for dV?

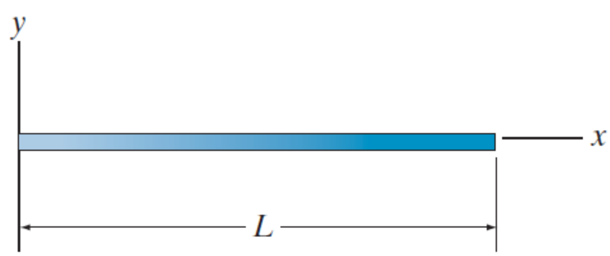
تمرینها:

۱- در شکل زیر میله در حال تعادل است. زاویه θ را بیابید.

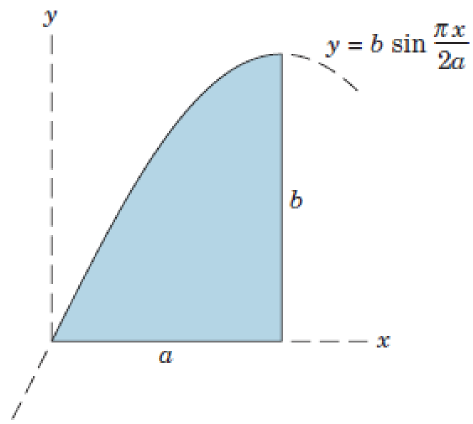


۲- در شکل زیر جرم بر واحد طول میله مطابق رابطه داده شده تغییر میکند. مکان مرکز جرم میله را بیابید.

$$m = m_0(1 + x^2/L^2)$$



۳- مرکز هندسی سطوح شکلهای زیر را بیابید.



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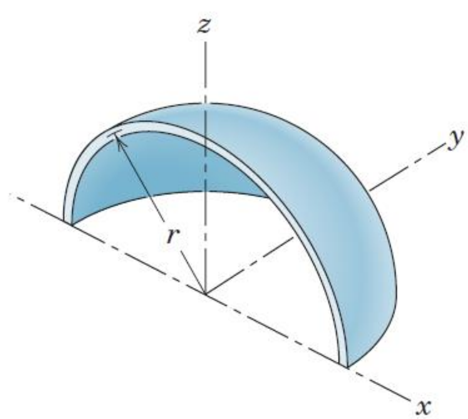

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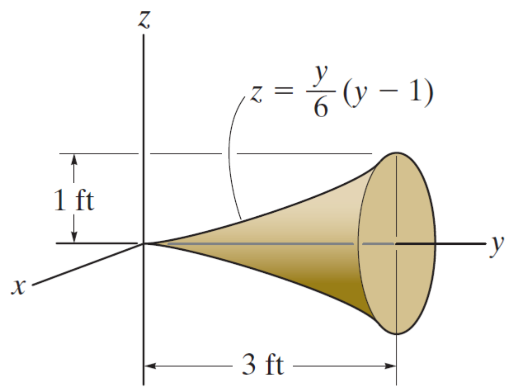
۴- مختصات مرکز حجم شکلهای زیر را بیابید.



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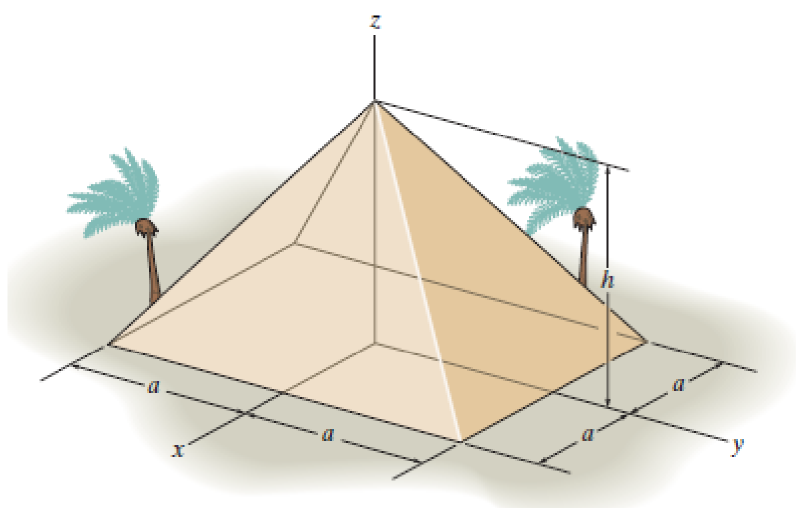

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موفق باشید.